Analytical modeling in astrophysics: Why and how?

Jean-Baptiste Durrive, IRAP

In collaboration with: P. Lesaffre, F. Boulanger, K. Ferrière, F. Rincon

P. Lesaffre, JB. Durrive, J. Goossaert, S. Poirier, S. Colombi, P. Richard, E. Allys, W. Bethune

Accepted in A&A, arxiv:2506.23659

General aim: Build diagnosis tools

Equations of motion (**physical parameters**)

↓

Velocity, density, magnetic fields

↓

Statistical features

↓

Observations

Outline

1) Context, motivation, goals

2) Turbulence model 1 (effective physical parameters) (‘BxC’)

3) Turbulence model 2 ($M \ll 1$, incompressible limit) (‘Muscats’)

4) Turbulence model 3 ($M \gg 1$, compressible limit) (work in progress)

5) Prospects & questionings

1) Context, motivation, goals

2) Turbulence model 1 (effective physical parameters) (‘BxC’)

3) Turbulence model 2 (M << 1, incompressible limit) (‘Muscats’)

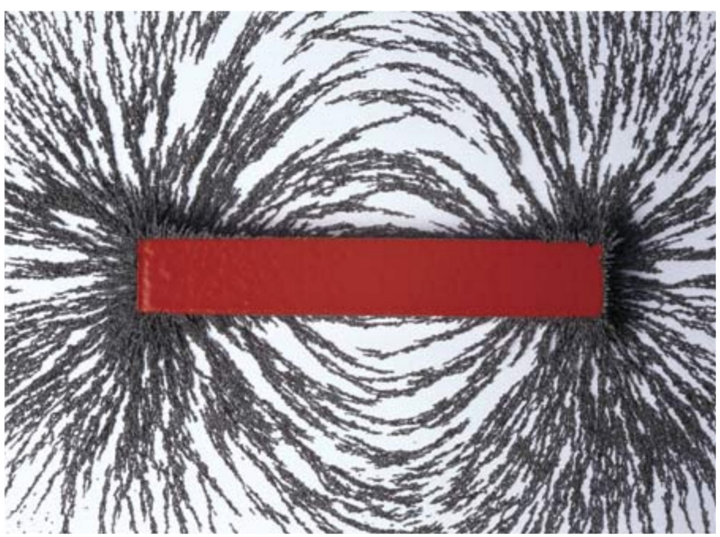
4) Turbulence model 3 (M >> 1, compressible limit) (work in progress)

5) Prospects & questionings

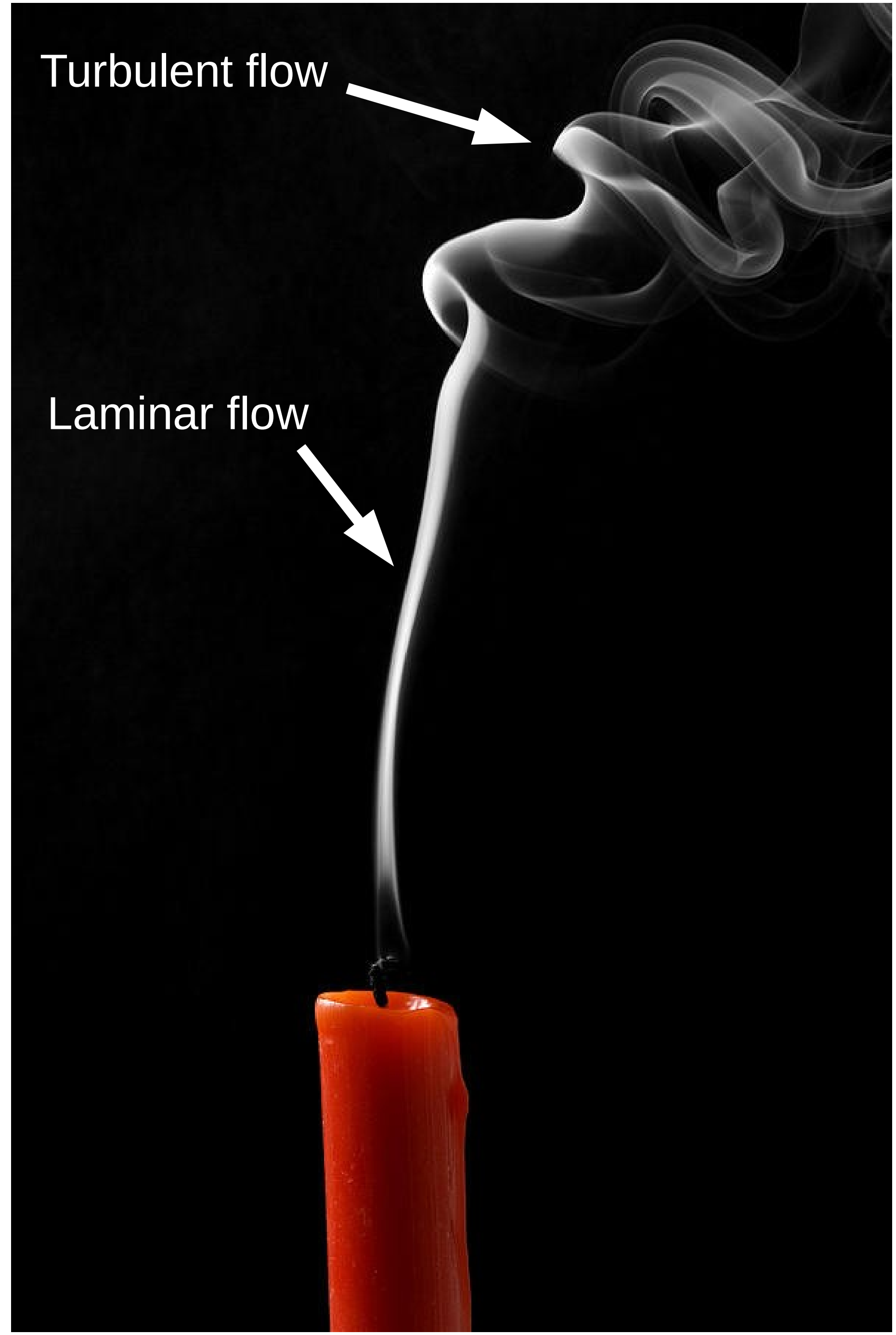
Physical processes I focus on

1) **Gravity** (which **can** be in any direction)

2) **Magnetic fields**

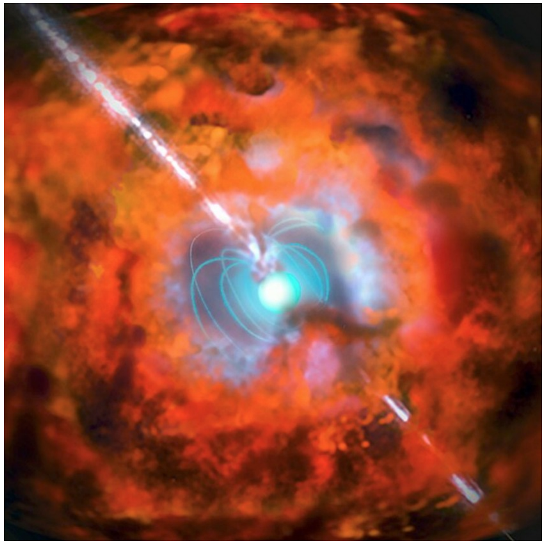


3) **Turbulence**



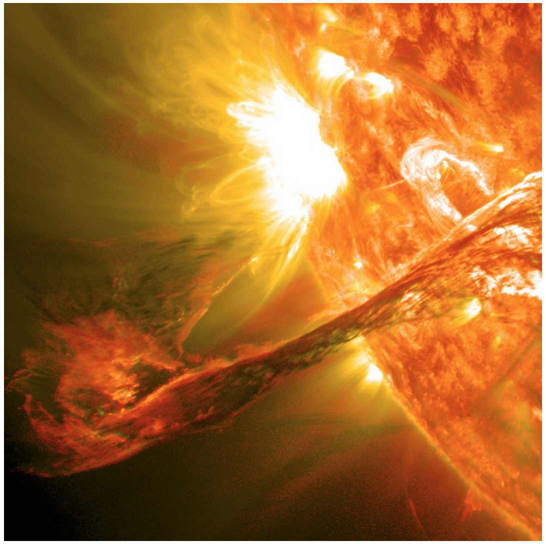
because they are ubiquitous in our Universe

Magnetars



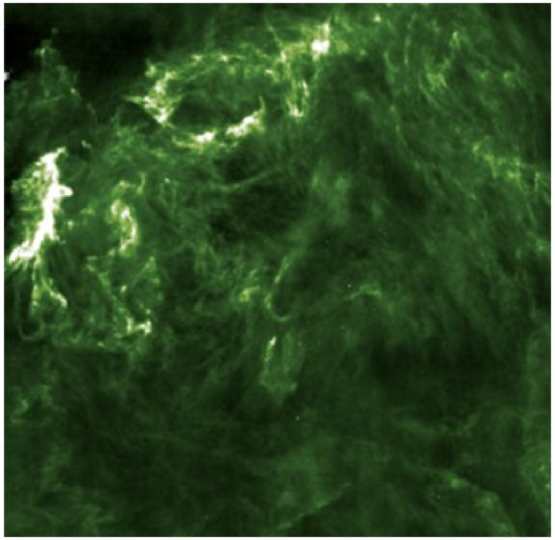
Artist work (Wikipedia)

Stars & planets



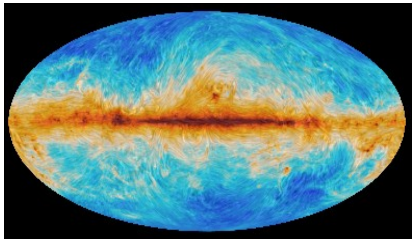
NASA, SDO 2012

Local interstellar medium



Miville-Deschênes+ 2010

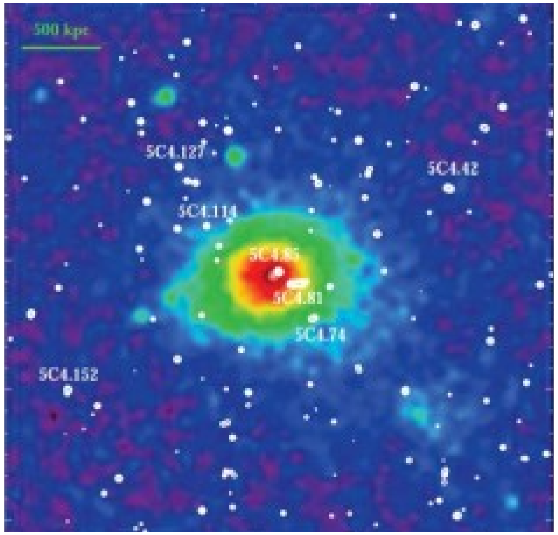
Galactic scales



ESA, Planck

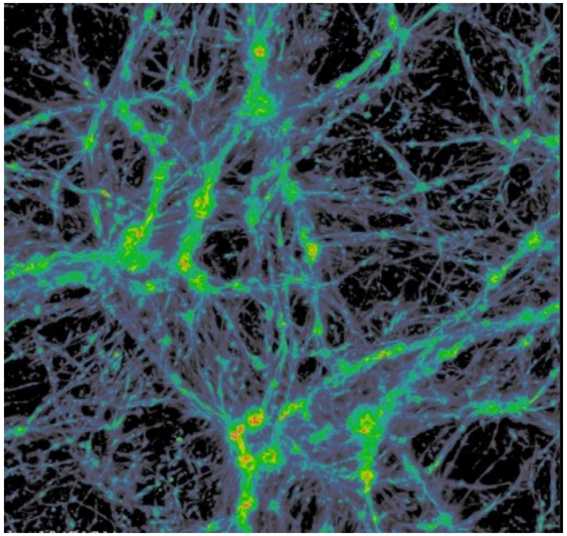
Fletcher et al 2011

Galaxy clusters



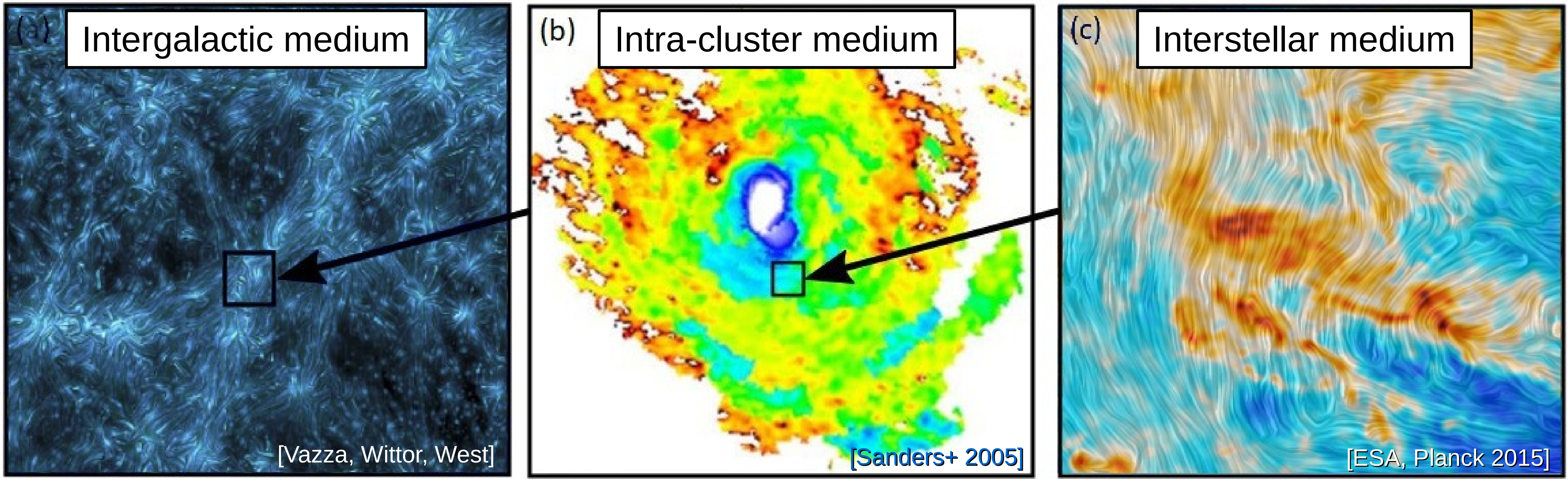
Bonafede et al 2010

Cosmic web, intergalactic medium



Vazza et al 2014

Astrophysical fluids I focus on



Very different media, but with a lot of connections and similarities!

Analytical : building alternatives to numerical simulations, which are very expensive

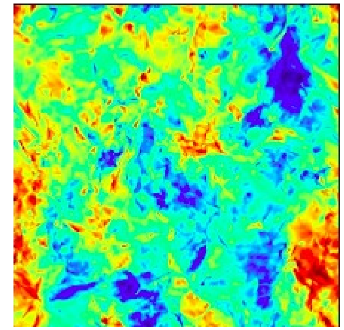
→ **turbulence synthesis**

Continuation of project BxB (PI: F. Boulanger)

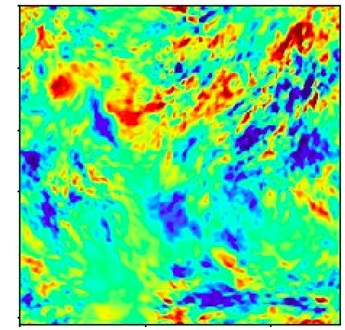
Model turbulent B fields of the ISM for:

1) Foreground **removal**:
   looking for primordial B-modes
2) Foreground **analysis**:
   looking for galactic **Astrophysics**

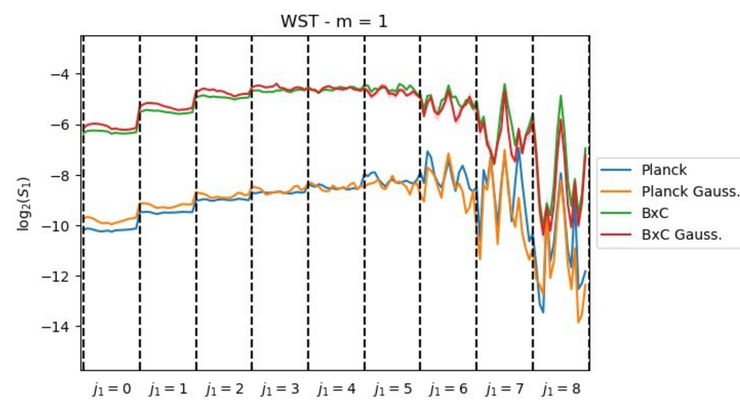
Data

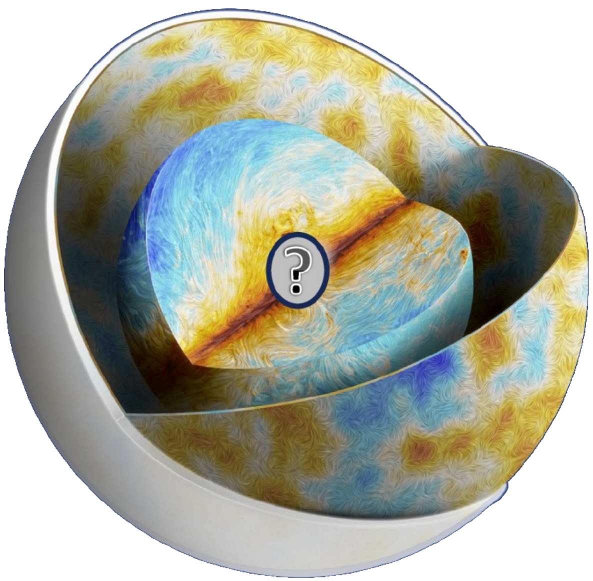


Model



Statistics





'French road map to CMB science' 2016

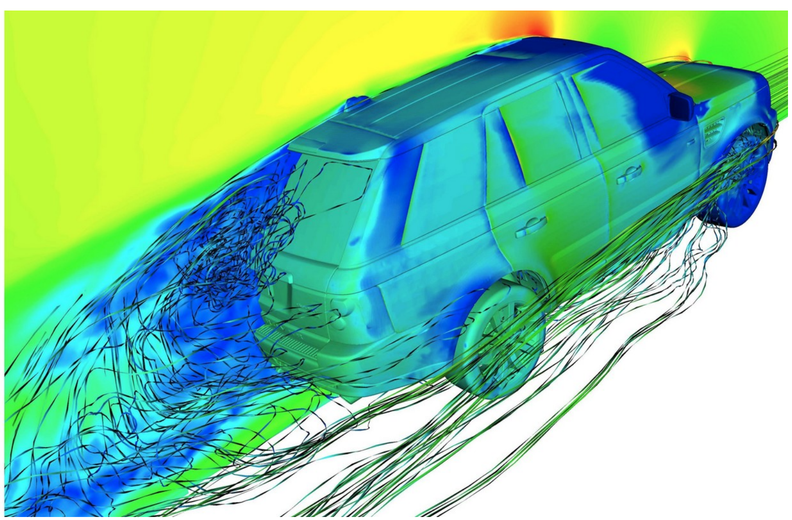
Build **physics**-based **parametric** models
(Mach, Reynolds, etc to compare to data)
→ **analytic** approach:
"syntheses" not simulations

Physics-based **parametric** models ?

<u>Fluid motion: intuitive description</u>

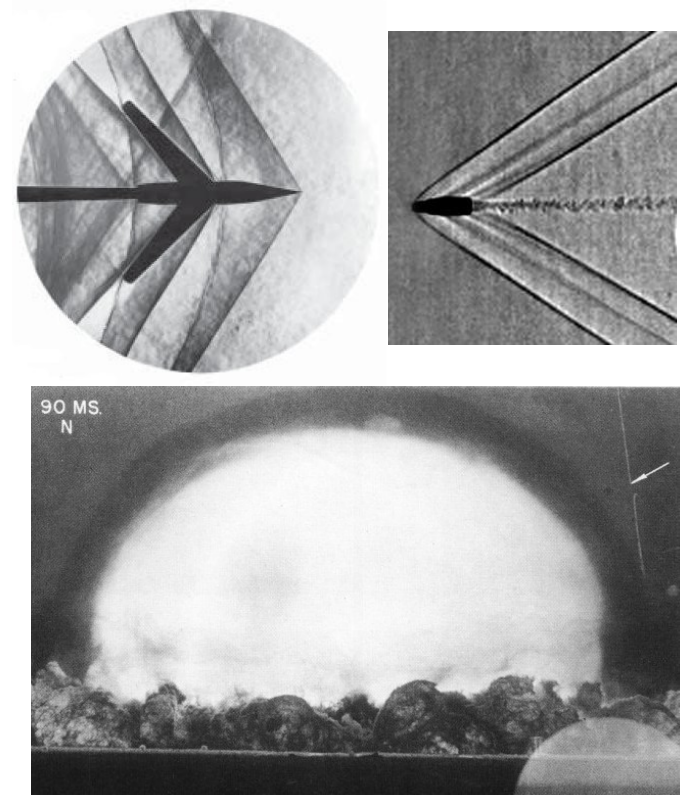
Honey: high **viscosity**, laminar

Air: low **viscosity**, turbulent



Water: low **compressibility**

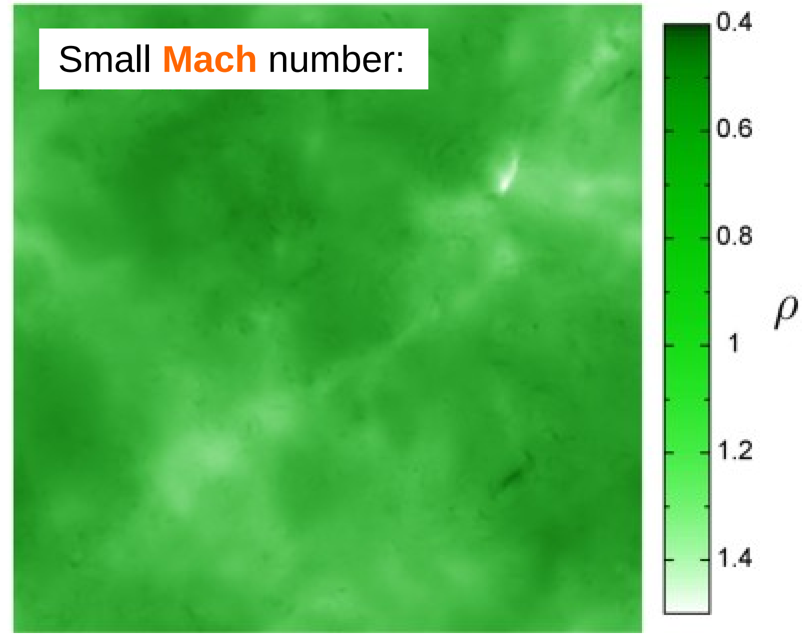
Air: high **compressibility**, shocks

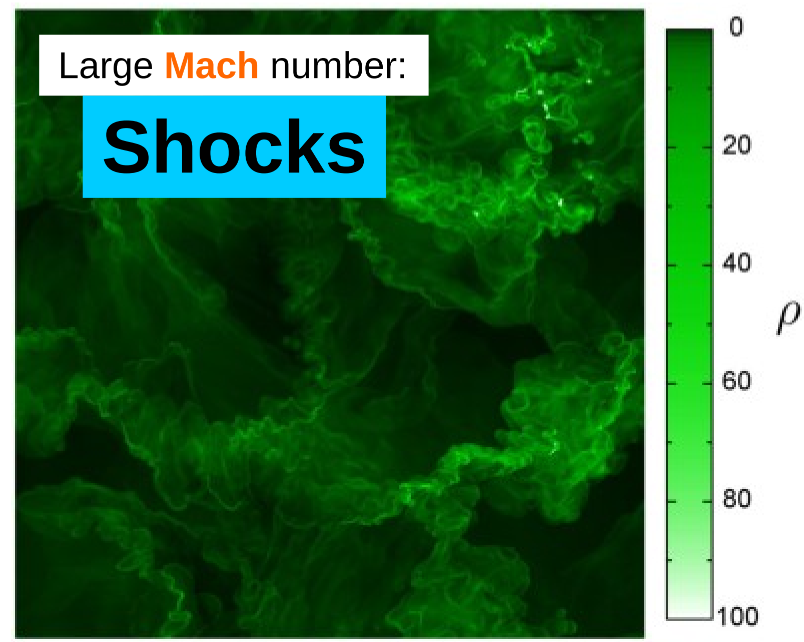


<u>Fluid motion: technical description</u>

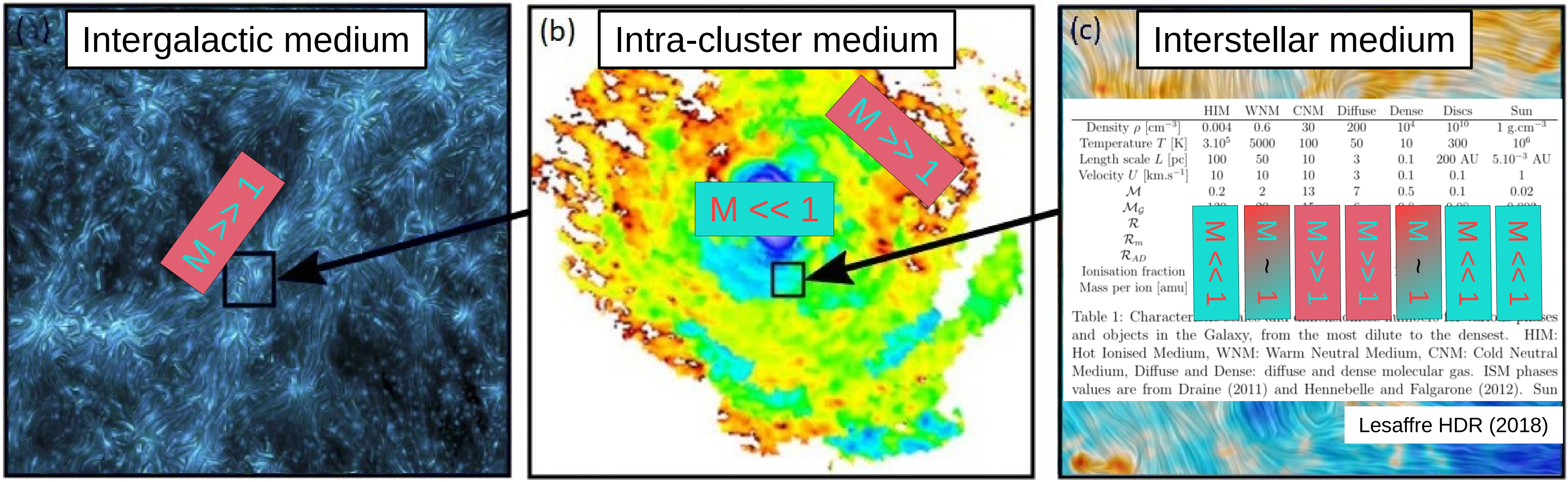
Small **Reynolds** number

Large **Reynolds** number





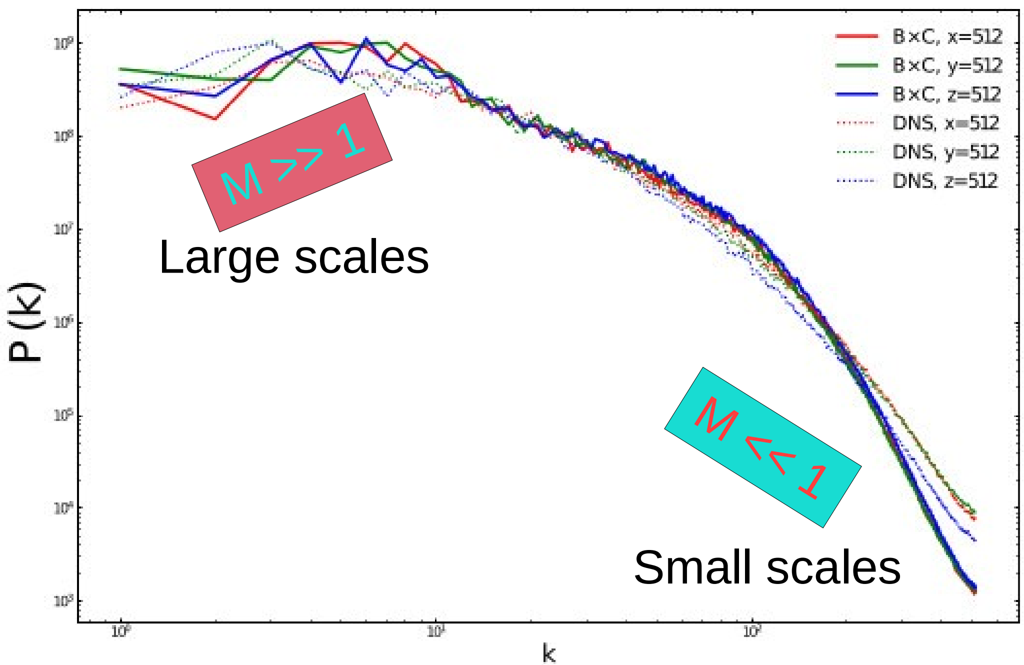
Konstantin+2016



	HIM	WNM	CNM	Diffuse	Dense	Discs	Sun
Density ρ [cm^{-3}]	0.004	0.6	30	200	10^4	10^{10}	1 g.cm^{-3}
Temperature T [K]	3.10^5	5000	100	50	10	300	10^6
Length scale L [pc]	100	50	10	3	0.1	200 AU	5.10^{-3} AU
Velocity U [km.s^{-1}]	10	10	10	3	0.1	0.1	1
$\mathcal{M}$	0.2	2	13	7	0.5	0.1	0.02
$\mathcal{M}_\mathcal{G}$							
$\mathcal{R}$							
$\mathcal{R}_m$							
$\mathcal{R}_{AD}$							
Ionisation fraction							
Mass per ion [amu]							

Table 1: Character[...] and objects in the Galaxy, from the most dilute to the densest. HIM: Hot Ionised Medium, WNM: Warm Neutral Medium, CNM: Cold Neutral Medium, Diffuse and Dense: diffuse and dense molecular gas. ISM phases values are from Draine (2011) and Hennebelle and Falgarone (2012). Sun

But can also depend on scales



Turbulence synthesis:

- random fields containing **intuitive/physical** free parameters (e.g. dissipation & injection scales)
- requiring **little computing resources** (CPU and time)
- **cheap, low-carbon alternative** to numerical simulations (we don't solve full sets of equations)

→ can be used to build **quicky** synthetic data (turbulent B fields), with **controllable statistics**

Possible applications to astrophysics/cosmology:

- Modeling galactic foregrounds

- Statistical characterization of interstellar/intracluster/intergalactic turbulence

- Dealing with intermittency (e.g. Cosmic ray propagation) [Maci et al 2025, Martin et al 2025]

- Extrapolating data to unresolved scales (e.g. modeling rainfall, cf [Posadas et al, NPG, 2015])
  Understand spatially unresolved measurements [Zakardjian et al 2025]

- Perform cheap simulations: testing a data analysis code with fake but realistic turbulent fields
  + initialize direct simulations [Maci et al 2025]

1) Context, motivation, goals

2) Turbulence model 1 (effective physical parameters) (‘BxC’)

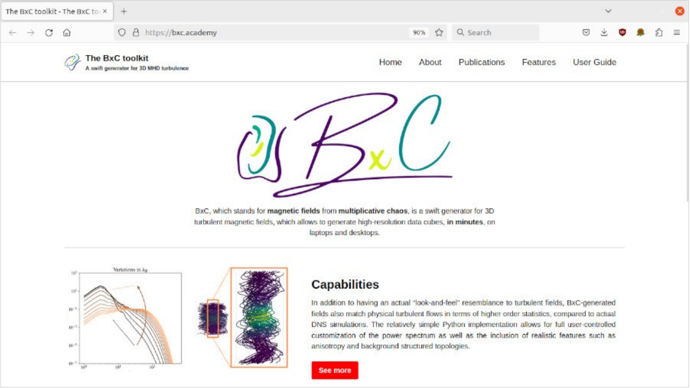
3) Turbulence model 2 (M << 1, incompressible limit) (‘Muscats’)

4) Turbulence model 3 (M >> 1, compressible limit) (work in progress)

5) Prospects & questionings

A practical result: BxC 'toolkit'

BxC website



Presentation:
https://bxc.academy

Papers:
Durrive et al, MNRAS (2020)
Durrive et al, PRE (2022)
Maci et al, ApJS (2024)
Maci et al, JPhys:CS (2025)

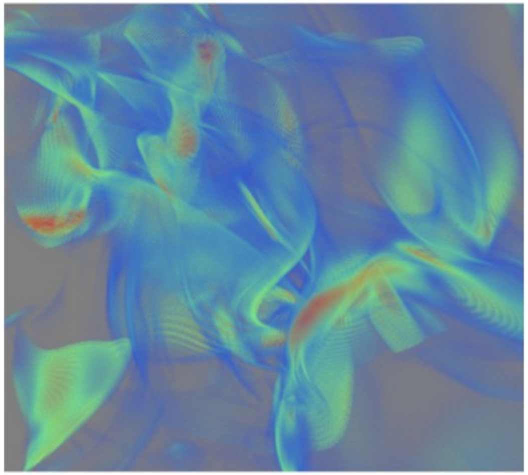
Download:
git clone https://github.com/danielamaci/bxc.github.io

Run the code:
python BxC.py

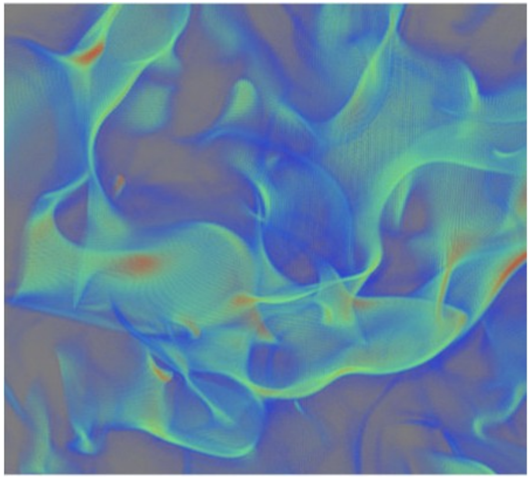
- a 3D vector field
- divergence-free
- with current sheets (curl of B)

- controllable power spectrum
- very low resources required (~ 1000 x cheaper than DNS)
- compact & intuitive analytical construction

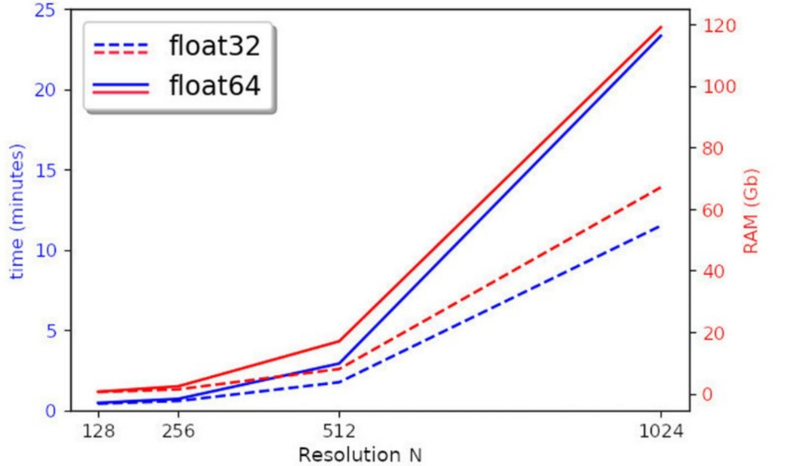
Simulation



Synthesis



BxC computing resources (in 3D)



Controllable power-spectrum

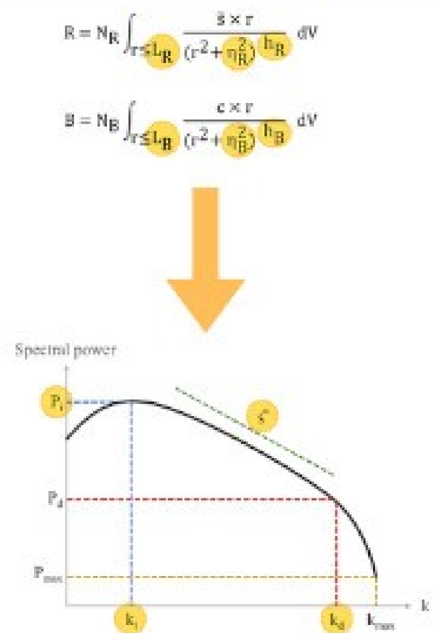
For details see Maci et al ApJS (2024)

https://bxc.academy/user_guide/ 90% Search

Here you can find the set of relations telling how the power spectrum varies as the user varies the parameters of the code. For more details on the parameter study that has been conducted see the article.

Relations
$k_i(L_B) \approx -68.5 L_B + 16.4$
$k_s(L_B) \approx \frac{1}{\sqrt{2\pi 0.03}} \exp\left(-\frac{(L_B-0.09)^2}{2(0.03)^2}\right)$
$k_d(\eta_B) \approx 6.2 \times 10^5 {\eta_B}^2 - 1.2 \times 10^4 \eta_B + 94$
$P_i(L_B;\bullet) \approx -42.6 {L_B}^2 + 35.3 L_B - 0.9$
$P_i(h_B;\bullet) \approx 1.5 \times 10^4 e^{-4.5 h_B}$
$P_i(\eta_B;\bullet) \approx 6.2 \times 10^4 {\eta_B}^2 + 18 \eta_B + 0.5$
$\zeta(h_B;\bullet) \approx -0.9 {h_B}^2 + 6.4 h_B - 10.1$
$\zeta(\eta_B;\bullet) \approx -201.5 \eta_B - 0.5$
$\zeta(h_B,\eta_B;\bullet) \approx A(\eta_B) h_B^2 + B(\eta_B) h_B + C(\eta_B)$
$A(\eta_B) \approx -4.6 \times 10^3 {\eta_B}^2 + 1.13 \times 10^2 \eta_B - 1.2$
$B(\eta_B) \approx 2.1 \times 10^4 {\eta_B}^2 - 5.2 \times 10^2 \eta_B + 7.7$
$C(\eta_B) \approx -2.2 \times 10^4 {\eta_B}^2 + 3.7 \times 10^2 \eta_B - 11$

Schematic representation



The notation (par;*), where par = LB, hB, or ηB, is used when a feature does not depend on "par" only, but the other parameters on which it depends are kept constant to the reference value.

GOAL = link

physical parameters
(effective here)

to

statistics/observables, here the power spectrum (~ Fourier transform)

1) Context, motivation, goals

2) Turbulence model 1 (effective physical parameters) (‘BxC’)

3) Turbulence model 2 ($M \ll 1$, incompressible limit) (‘Muscats’)

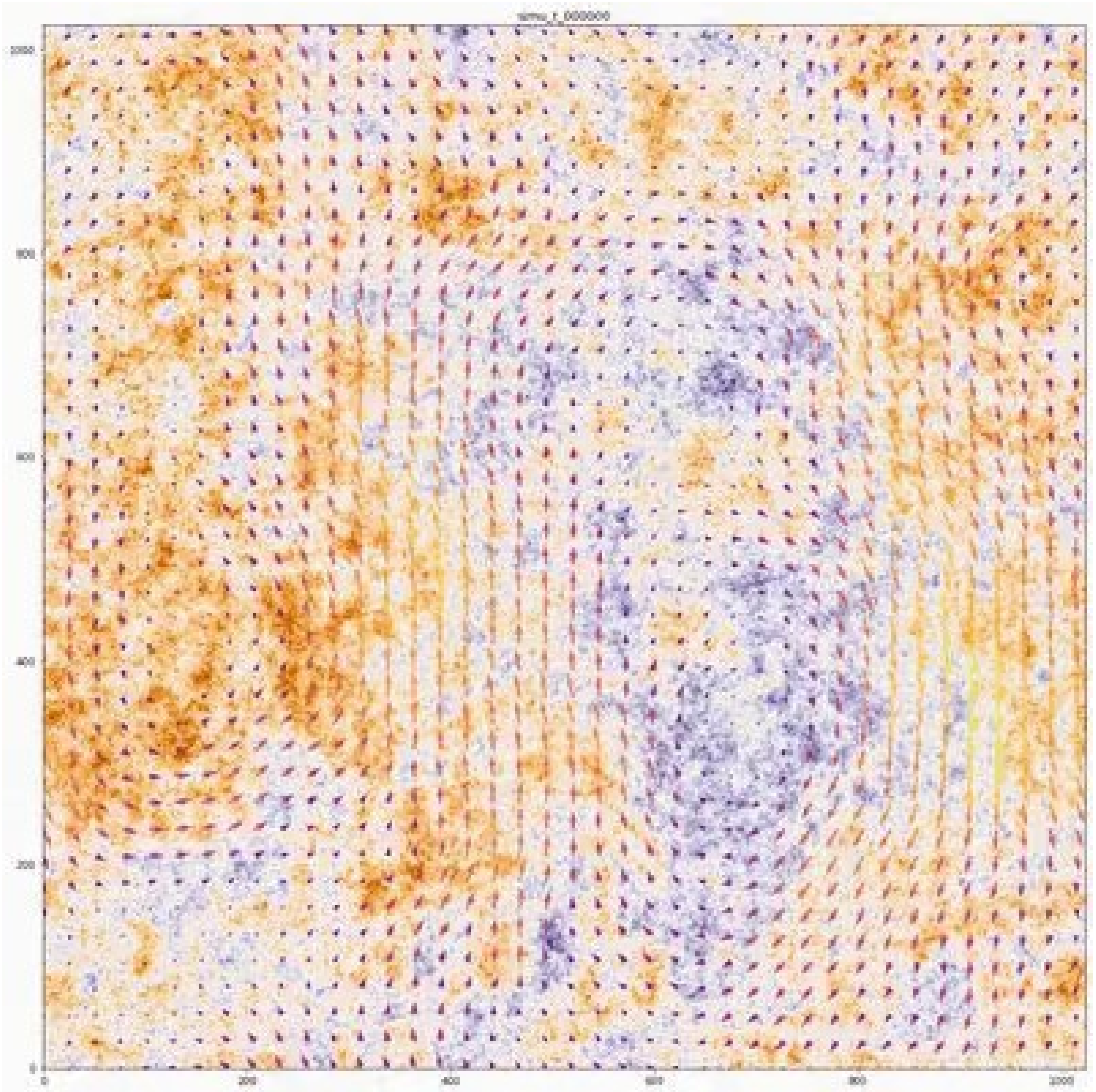
4) Turbulence model 3 ($M \gg 1$, compressible limit) (work in progress)

5) Prospects & questionings

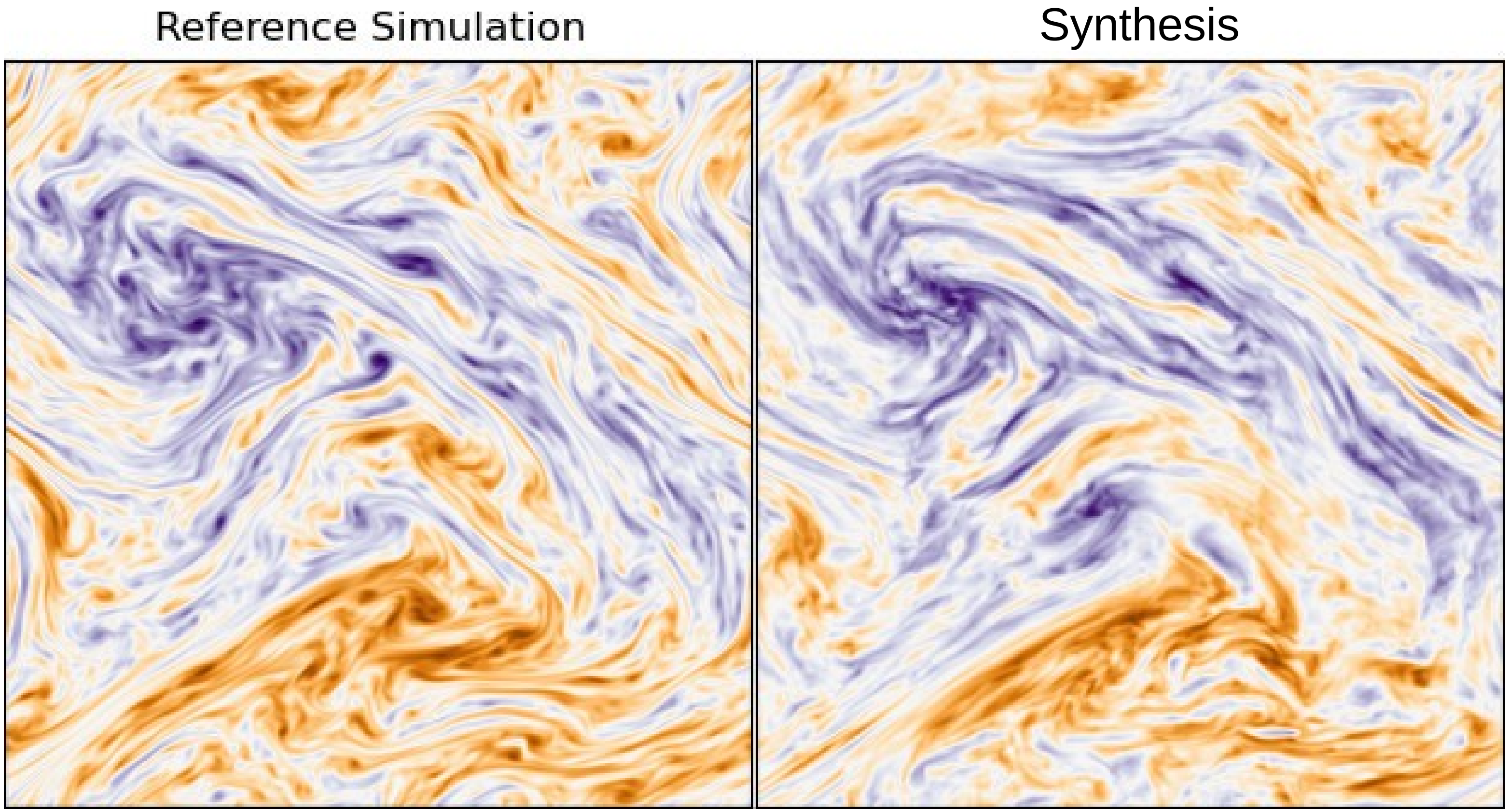
Multi-Scale Turbulence Synthesis (MUSCATS)

P. Lesaffre, JB. Durrive, J. Goossaert, S. Poirier, S. Colombi, P. Richard, E. Allys, W. Bethune

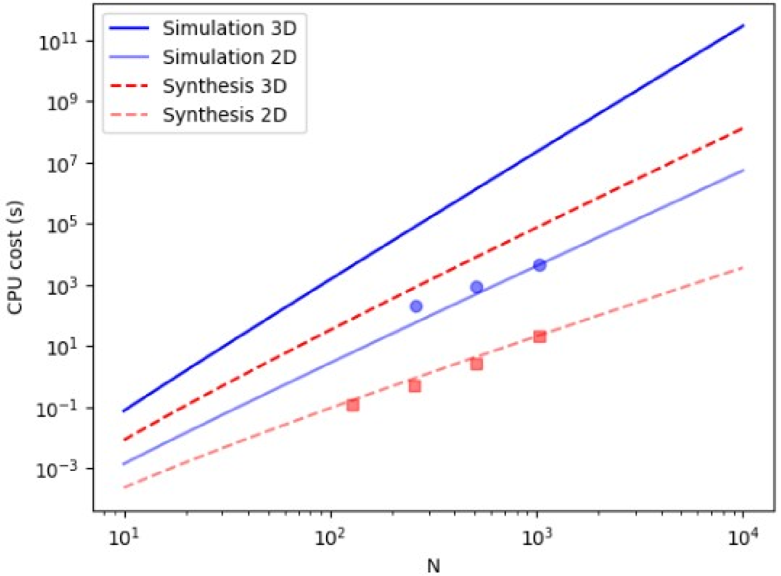
Accepted in A&A, Arxiv: 2506.23659



Multi-Scale Turbulence Synthesis (MUSCATS)

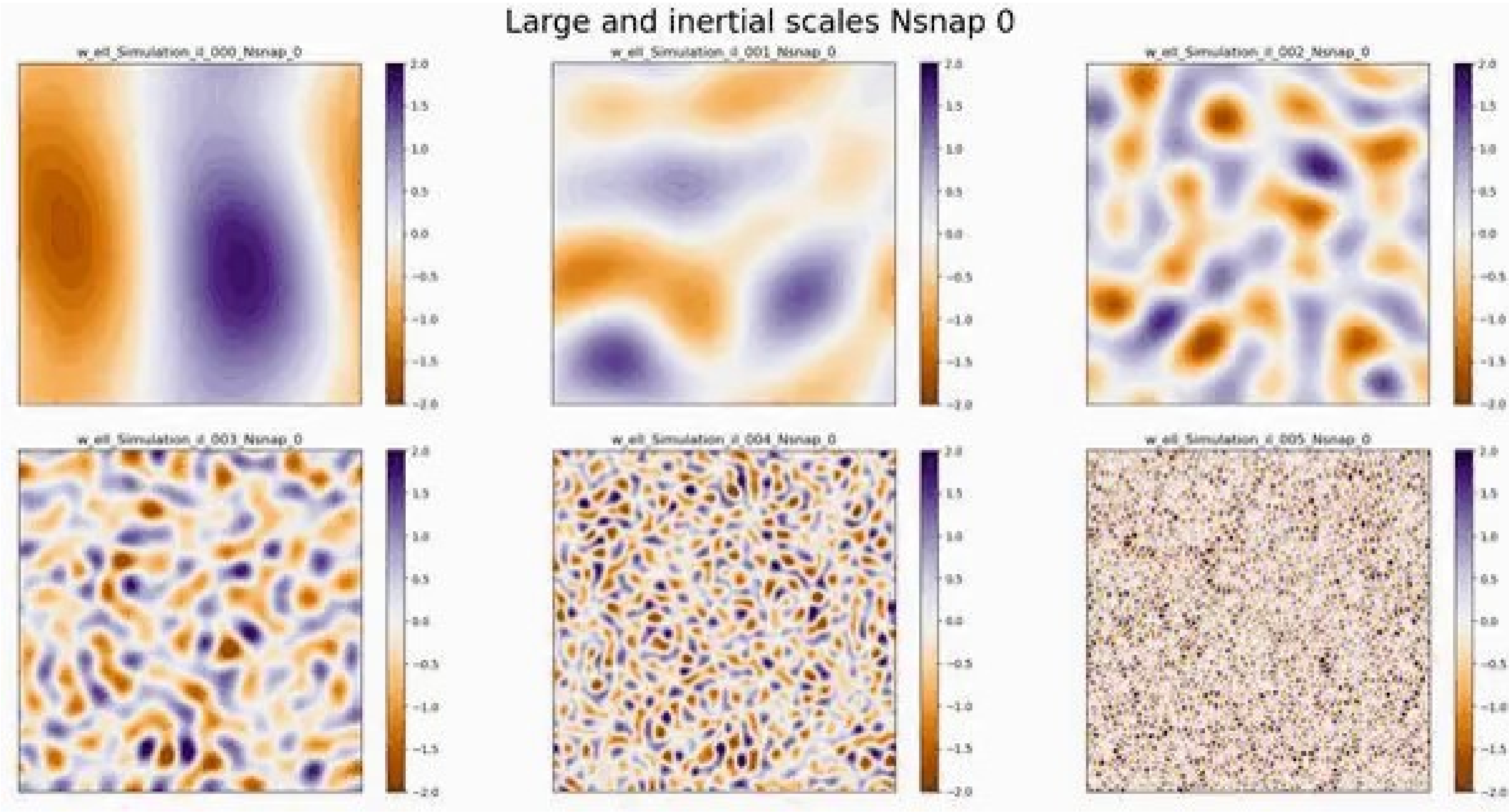


pixel-by-pixel similarity with DNS
(not only statistics)
and
with little resources
(cf plot on the right)
and
Physically based
(based on momentum conservation)



Multi-Scale Turbulence Synthesis (MUSCATS)

Key = study interaction between scales

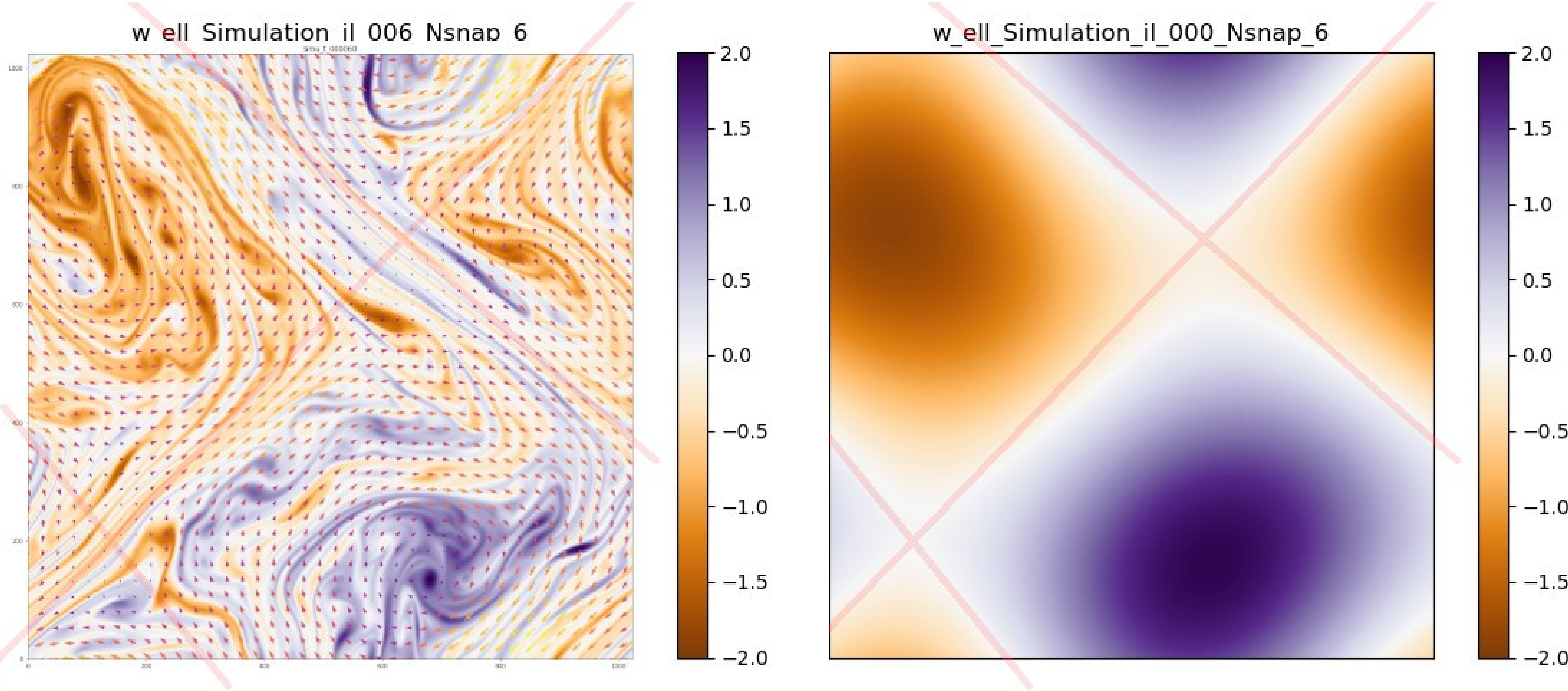


$$\partial_t \boldsymbol{W} + (\boldsymbol{v}[\boldsymbol{W}].\boldsymbol{\nabla})\boldsymbol{W} = \boldsymbol{S}[\boldsymbol{W}].\boldsymbol{W} + \boldsymbol{D}[\boldsymbol{W}]$$

$$\partial_t \tilde{\boldsymbol{W}}_\ell \simeq -\tilde{\boldsymbol{v}}_{\geqslant\ell}.\boldsymbol{\nabla}\tilde{\boldsymbol{W}}_\ell + \tilde{\boldsymbol{S}}_{\geqslant\ell}.\tilde{\boldsymbol{W}}_\ell + \boldsymbol{D}[\tilde{\boldsymbol{W}}_\ell]$$

Multi-Scale Turbulence Synthesis (MUSCATS)

All scales are highly correlated:

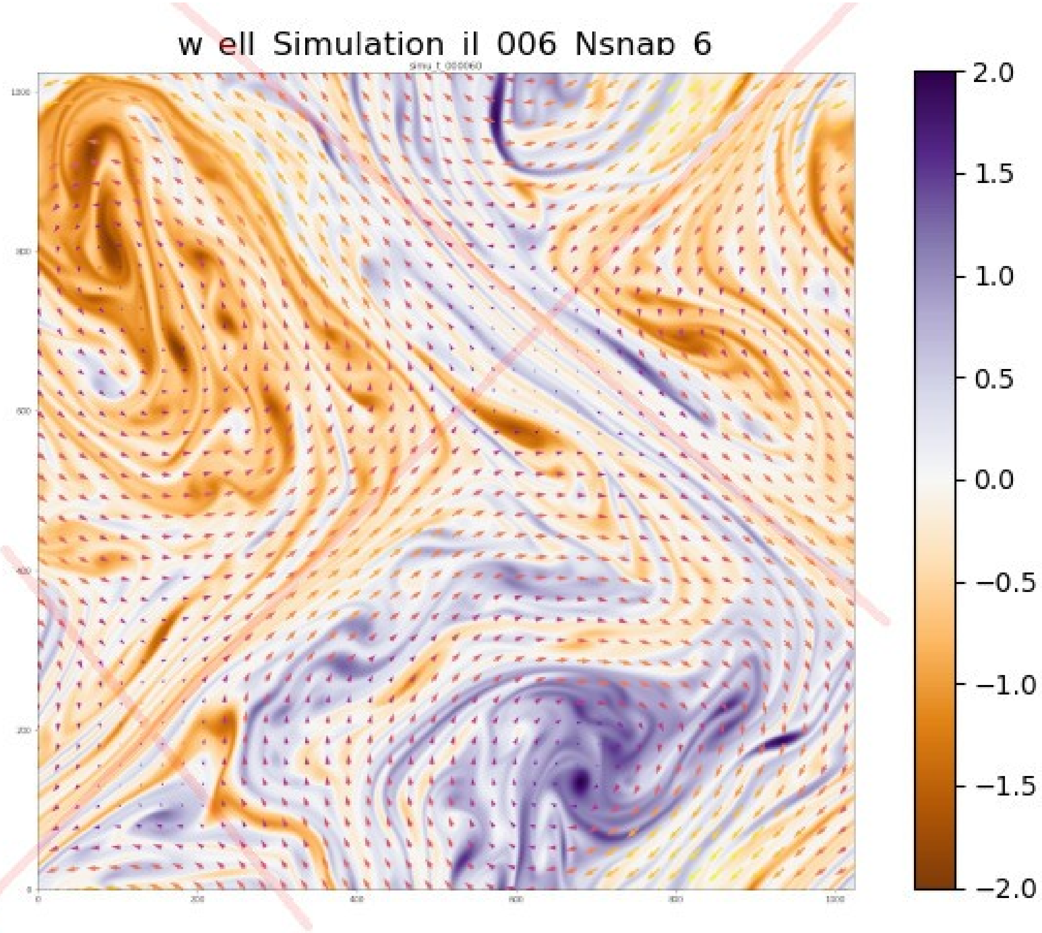


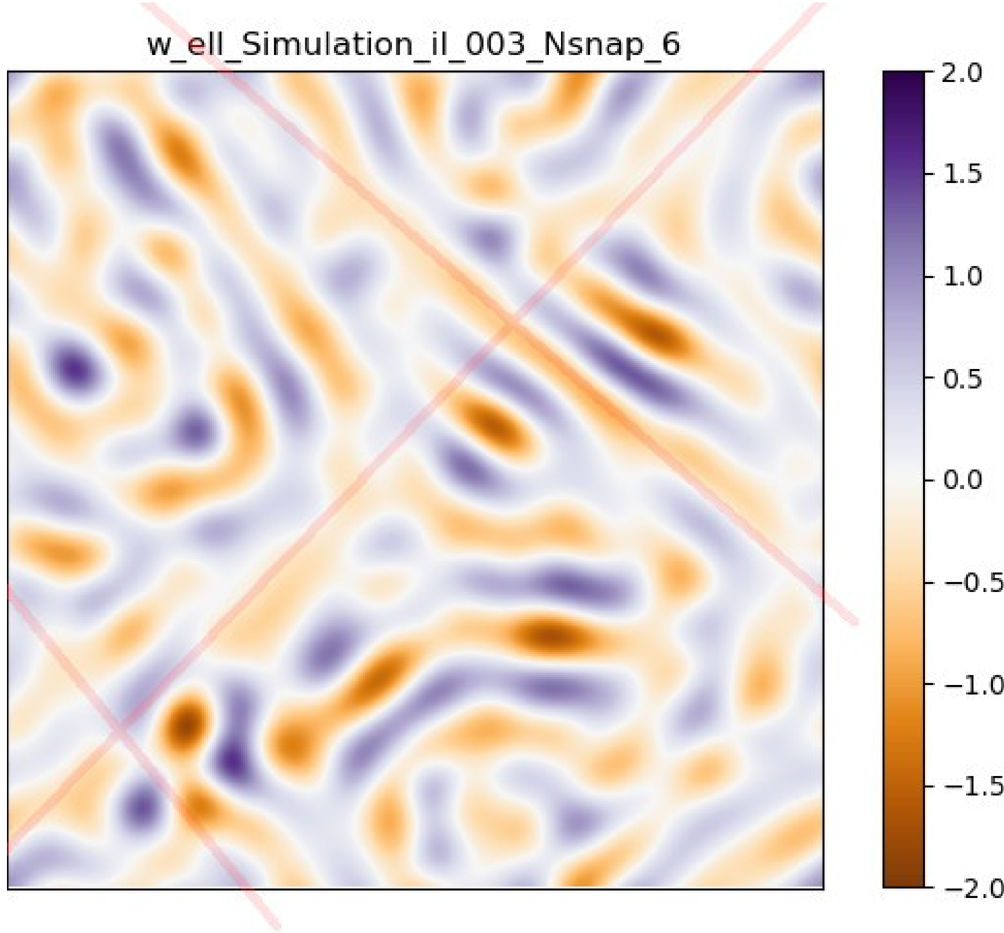
$$\partial_t \boldsymbol{W} + (\boldsymbol{v}[\boldsymbol{W}].\nabla)\boldsymbol{W} = \boldsymbol{S}[\boldsymbol{W}].\boldsymbol{W} + \boldsymbol{D}[\boldsymbol{W}]$$

$$\partial_t \tilde{\boldsymbol{W}}_\ell \simeq -\tilde{\boldsymbol{v}}_{\geqslant \ell}.\boldsymbol{\nabla}\tilde{\boldsymbol{W}}_\ell + \tilde{\boldsymbol{S}}_{\geqslant \ell}.\tilde{\boldsymbol{W}}_\ell + \boldsymbol{D}[\tilde{\boldsymbol{W}}_\ell]$$

Multi-Scale Turbulence Synthesis (MUSCATS)

All scales are highly correlated:



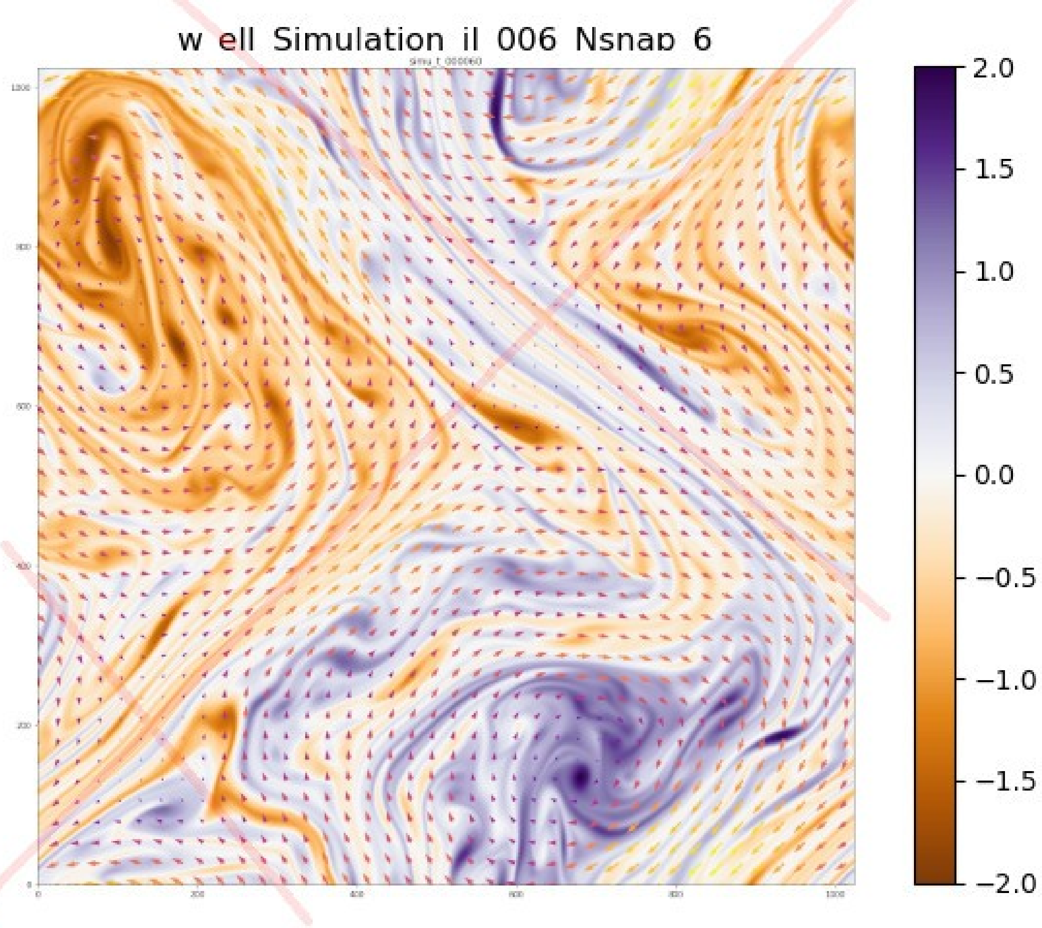


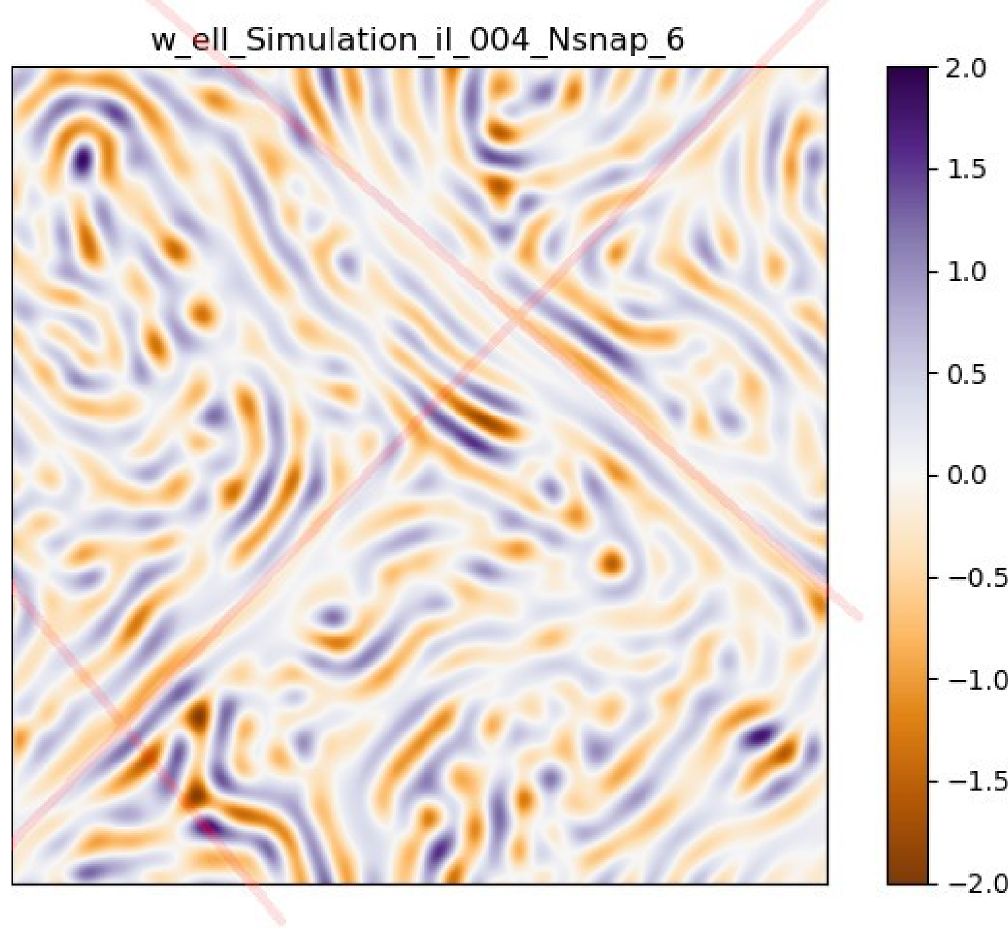
$$\partial_t \boldsymbol{W} + (\boldsymbol{v}[\boldsymbol{W}].\boldsymbol{\nabla})\boldsymbol{W} = \boldsymbol{S}[\boldsymbol{W}].\boldsymbol{W} + \boldsymbol{D}[\boldsymbol{W}]$$

$$\partial_t \tilde{\boldsymbol{W}}_\ell \simeq -\tilde{\boldsymbol{v}}_{\geqslant\ell}.\boldsymbol{\nabla}\tilde{\boldsymbol{W}}_\ell + \tilde{\boldsymbol{S}}_{\geqslant\ell}.\tilde{\boldsymbol{W}}_\ell + \boldsymbol{D}[\tilde{\boldsymbol{W}}_\ell]$$

Multi-Scale Turbulence Synthesis (MUSCATS)

All scales are highly correlated:



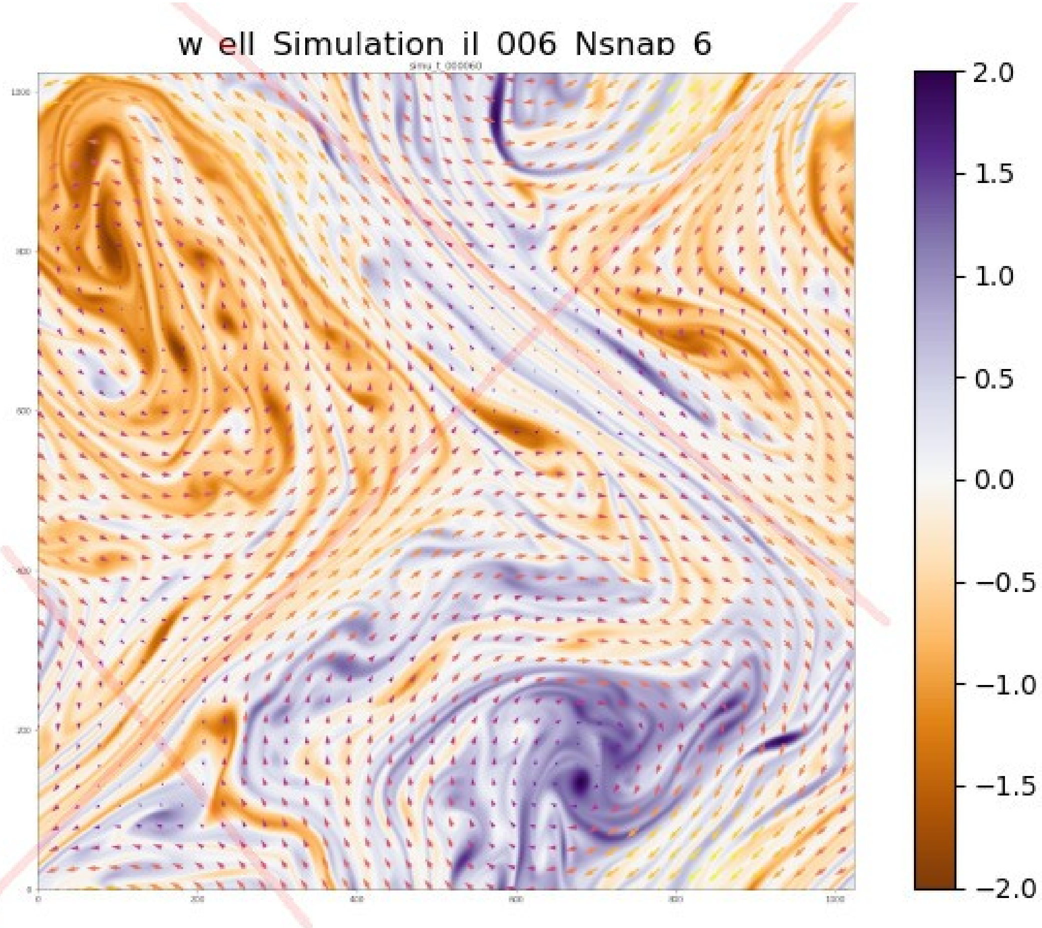


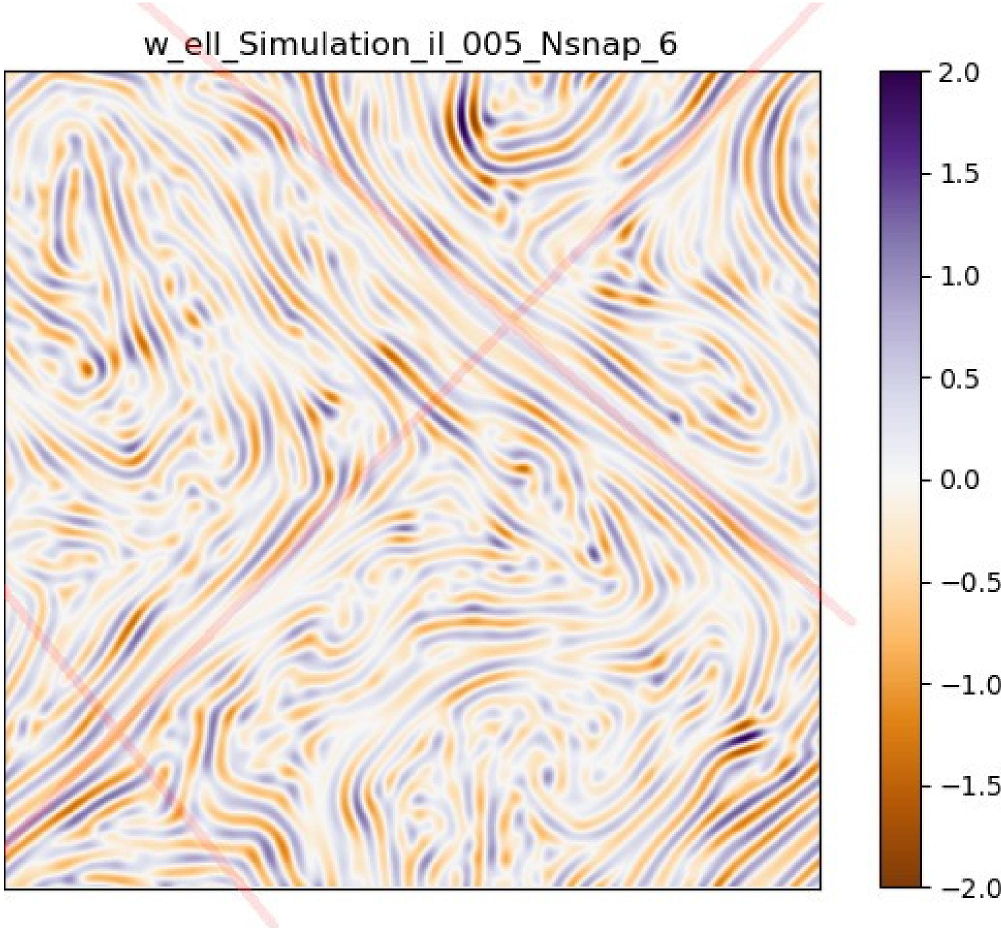
$$\partial_t \boldsymbol{W} + (\boldsymbol{v}[\boldsymbol{W}].\boldsymbol{\nabla})\boldsymbol{W} = \boldsymbol{S}[\boldsymbol{W}].\boldsymbol{W} + \boldsymbol{D}[\boldsymbol{W}]$$

$$\partial_t \tilde{\boldsymbol{W}}_\ell \simeq -\tilde{\boldsymbol{v}}_{\geqslant \ell}.\boldsymbol{\nabla}\tilde{\boldsymbol{W}}_\ell + \tilde{\boldsymbol{S}}_{\geqslant \ell}.\tilde{\boldsymbol{W}}_\ell + \boldsymbol{D}[\tilde{\boldsymbol{W}}_\ell]$$

Multi-Scale Turbulence Synthesis (MUSCATS)

All scales are highly correlated:





$$\partial_t \boldsymbol{W} + (\boldsymbol{v}[\boldsymbol{W}].\boldsymbol{\nabla})\boldsymbol{W} = \boldsymbol{S}[\boldsymbol{W}].\boldsymbol{W} + \boldsymbol{D}[\boldsymbol{W}]$$

$$\partial_t \tilde{\boldsymbol{W}}_\ell \simeq -\tilde{\boldsymbol{v}}_{\geqslant \ell}.\boldsymbol{\nabla}\tilde{\boldsymbol{W}}_\ell + \tilde{\boldsymbol{S}}_{\geqslant \ell}.\tilde{\boldsymbol{W}}_\ell + \boldsymbol{D}[\tilde{\boldsymbol{W}}_\ell]$$

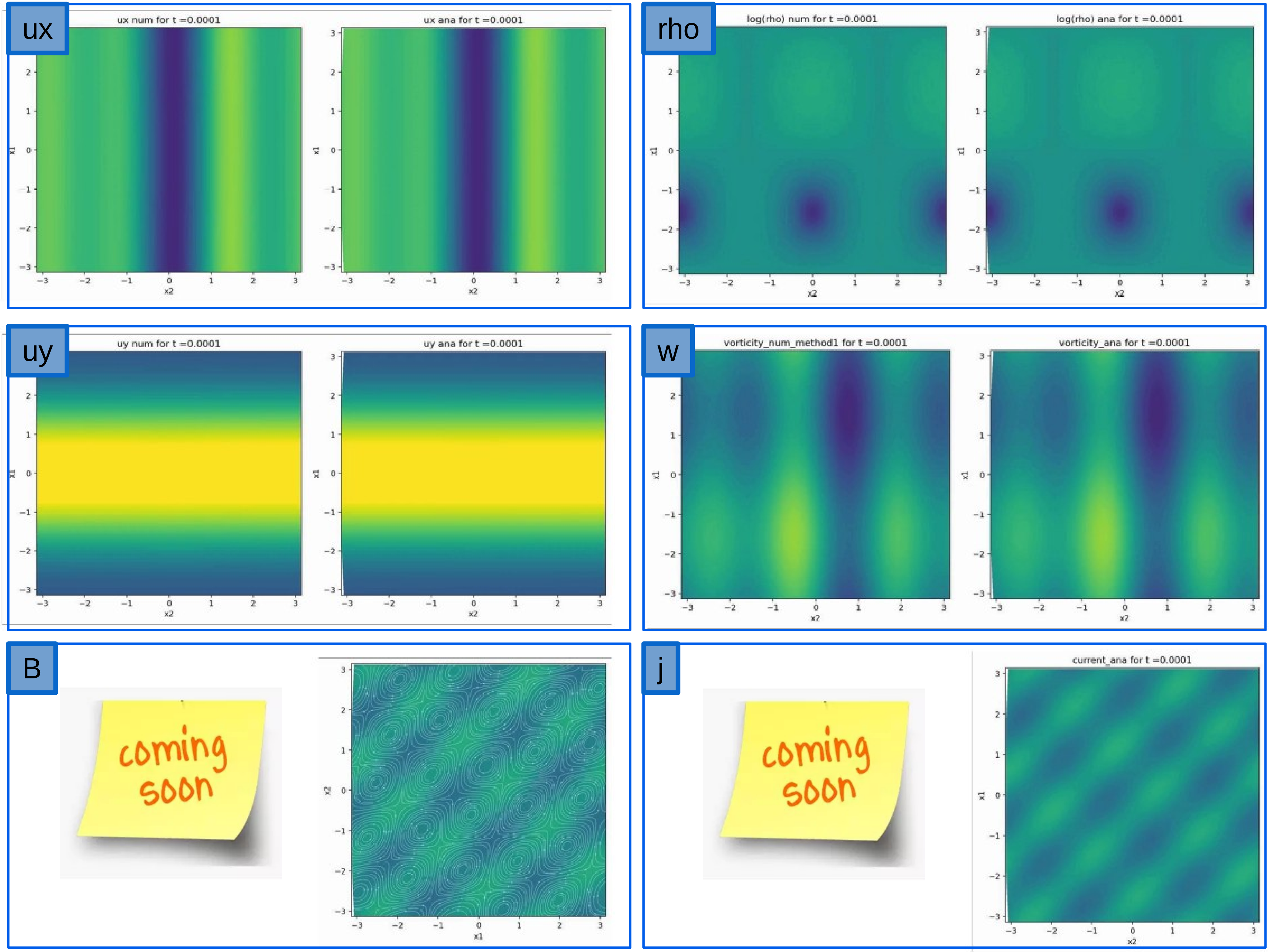
1) Context, motivation, goals

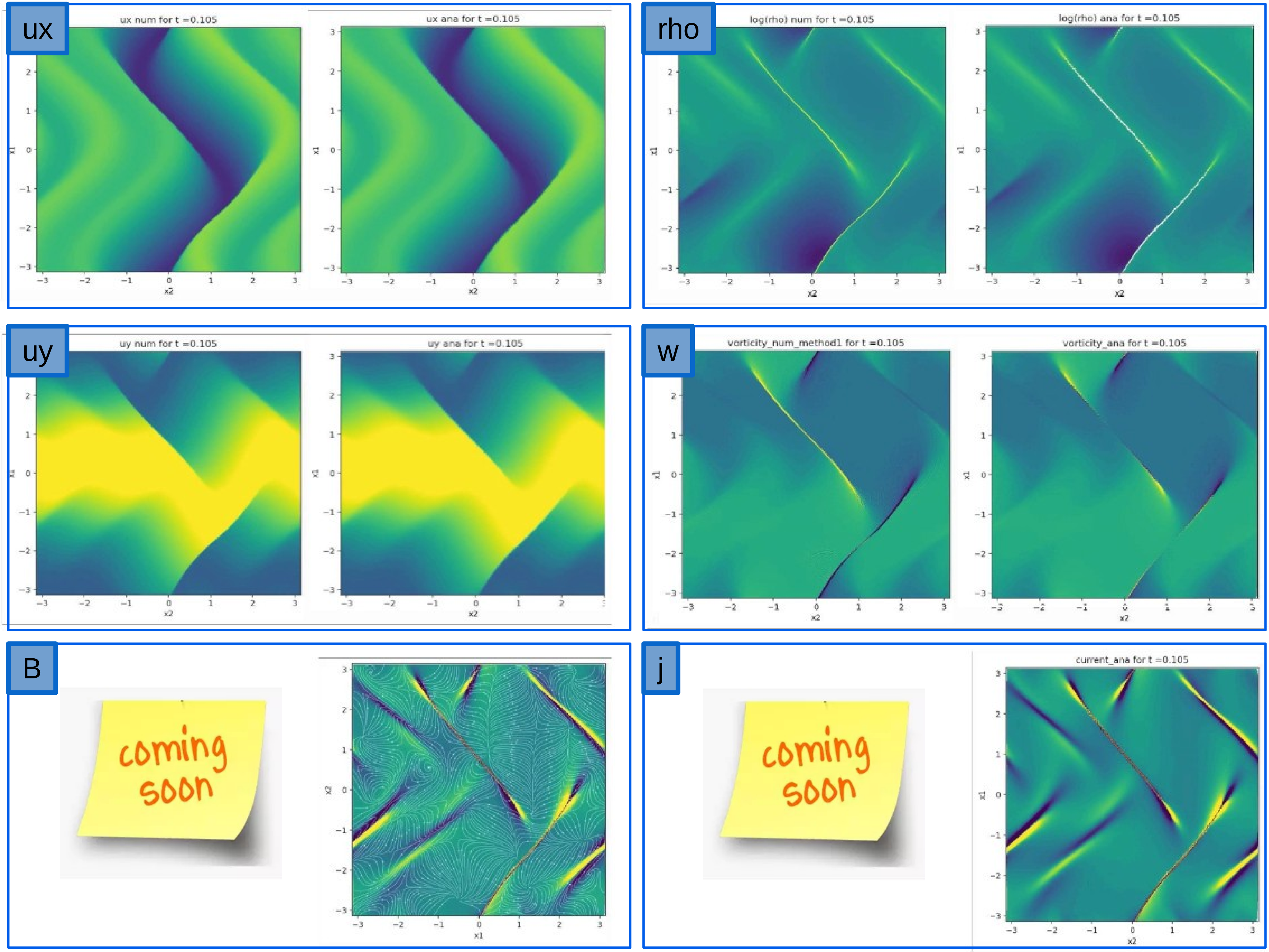
2) Turbulence model 1 (effective physical parameters) ('BxC')

3) Turbulence model 2 ($M \ll 1$, incompressible limit) ('Muscats')

4) Turbulence model 3 ($M \gg 1$, compressible limit) (work in progress)

5) Prospects & questionings

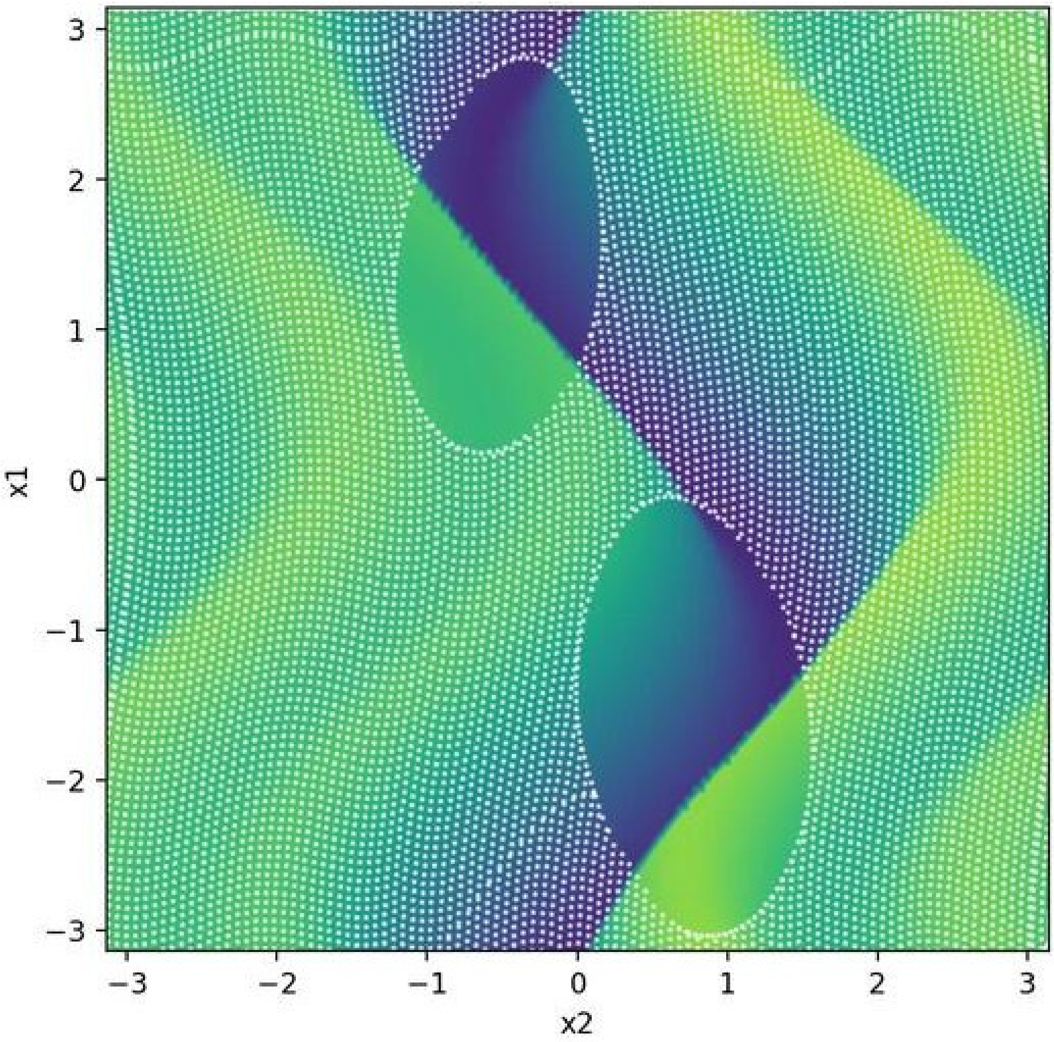

ux
ux num for t =0.0001
ux ana for t =0.0001
rho
log(rho) num for t =0.0001
log(rho) ana for t =0.0001
uy
uy num for t =0.0001
uy ana for t =0.0001
w
vorticity_num_method1 for t =0.0001
vorticity_ana for t =0.0001
B
coming soon
j
coming soon
current_ana for t =0.0001
x1
x2



ux
ux num for t =0.105
ux ana for t =0.105
rho
log(rho) num for t =0.105
log(rho) ana for t =0.105
uy
uy num for t =0.105
uy ana for t =0.105
w
vorticity_num_method1 for t =0.105
vorticity_ana for t =0.105
B
coming soon
j
coming soon
current_ana for t =0.105
x1
x2

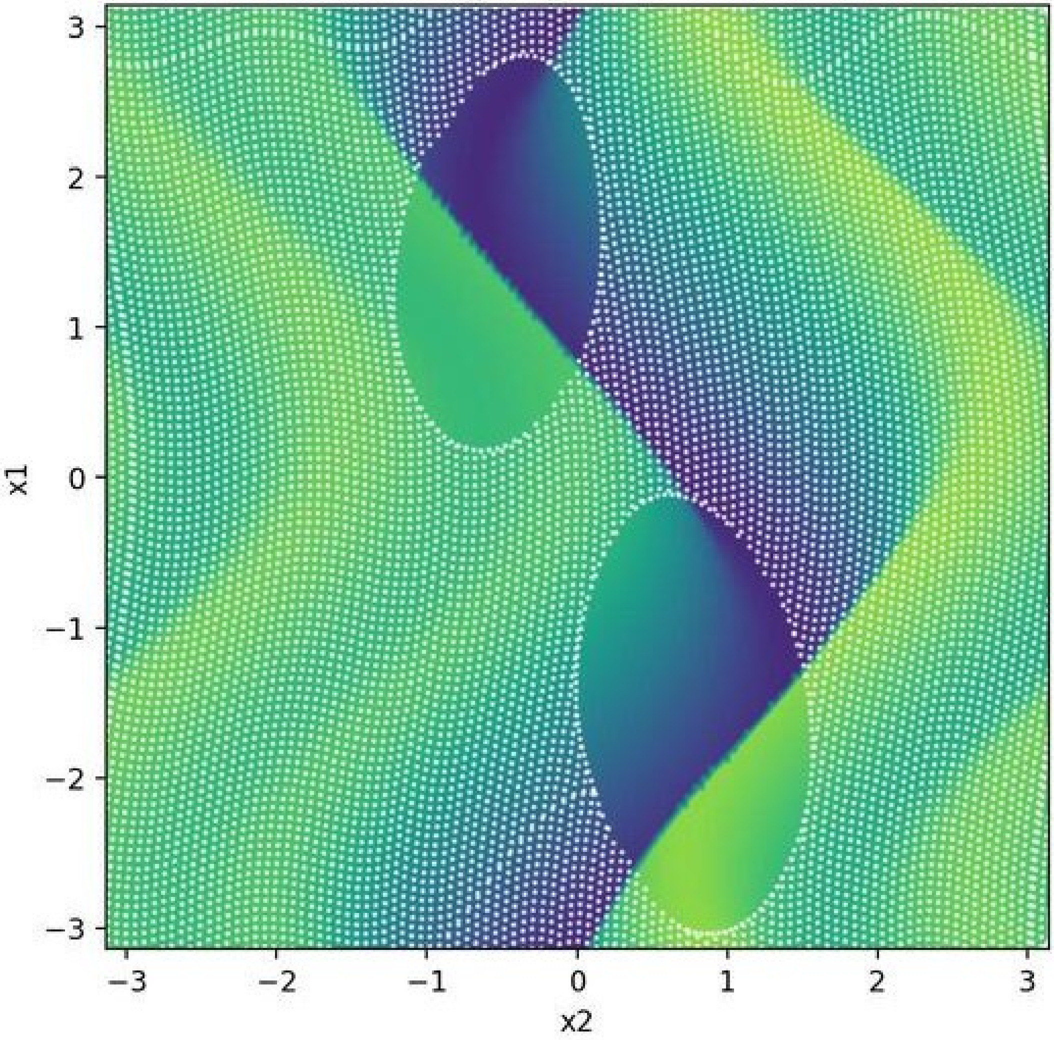
Key point: remove fluid elements that will end up in shocks

Initial positions

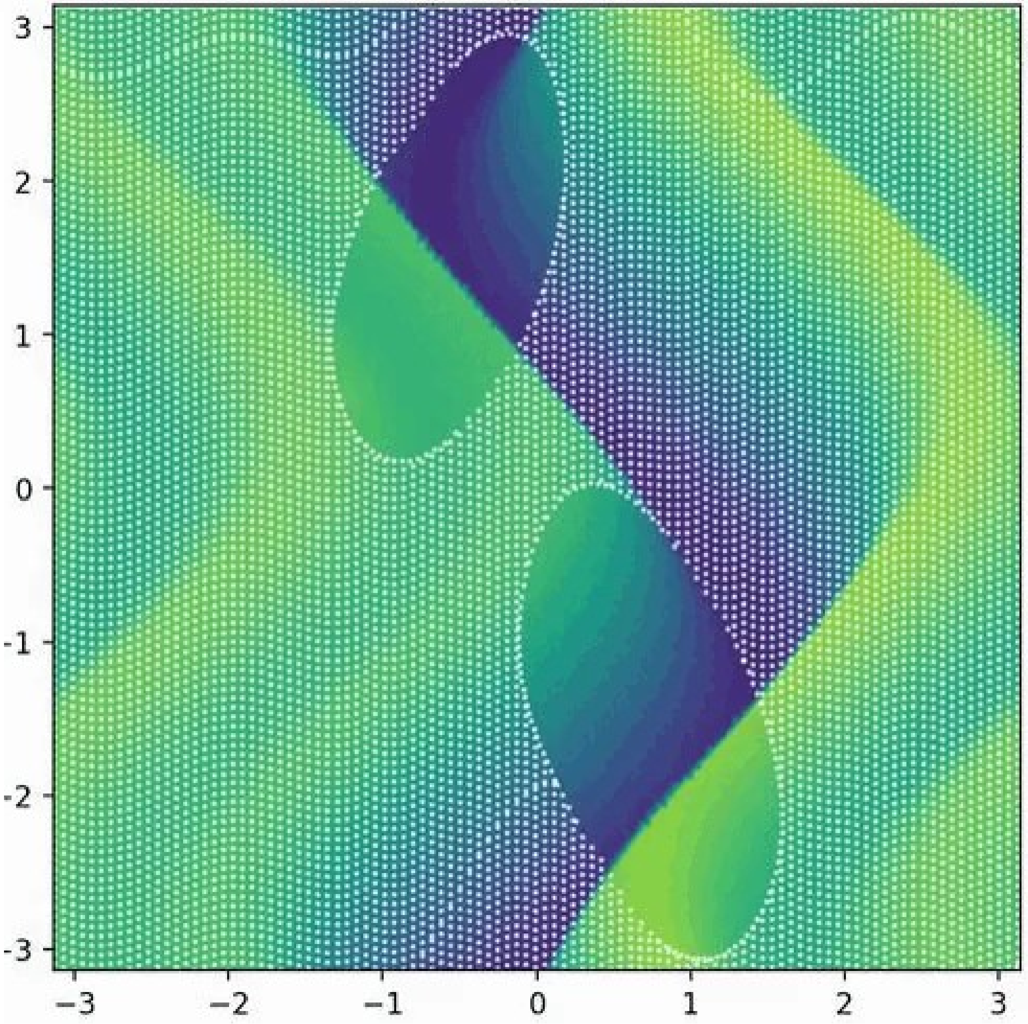


Key point: remove fluid elements that will end up in shocks

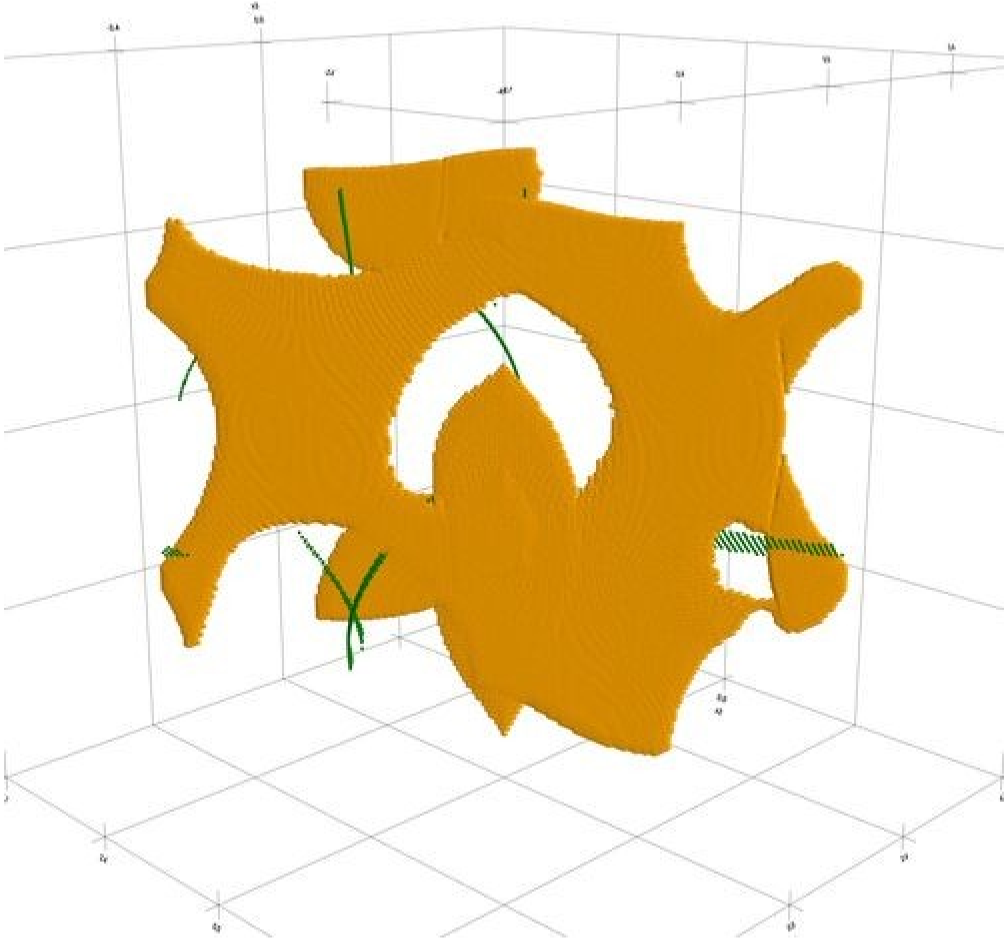
Initial positions



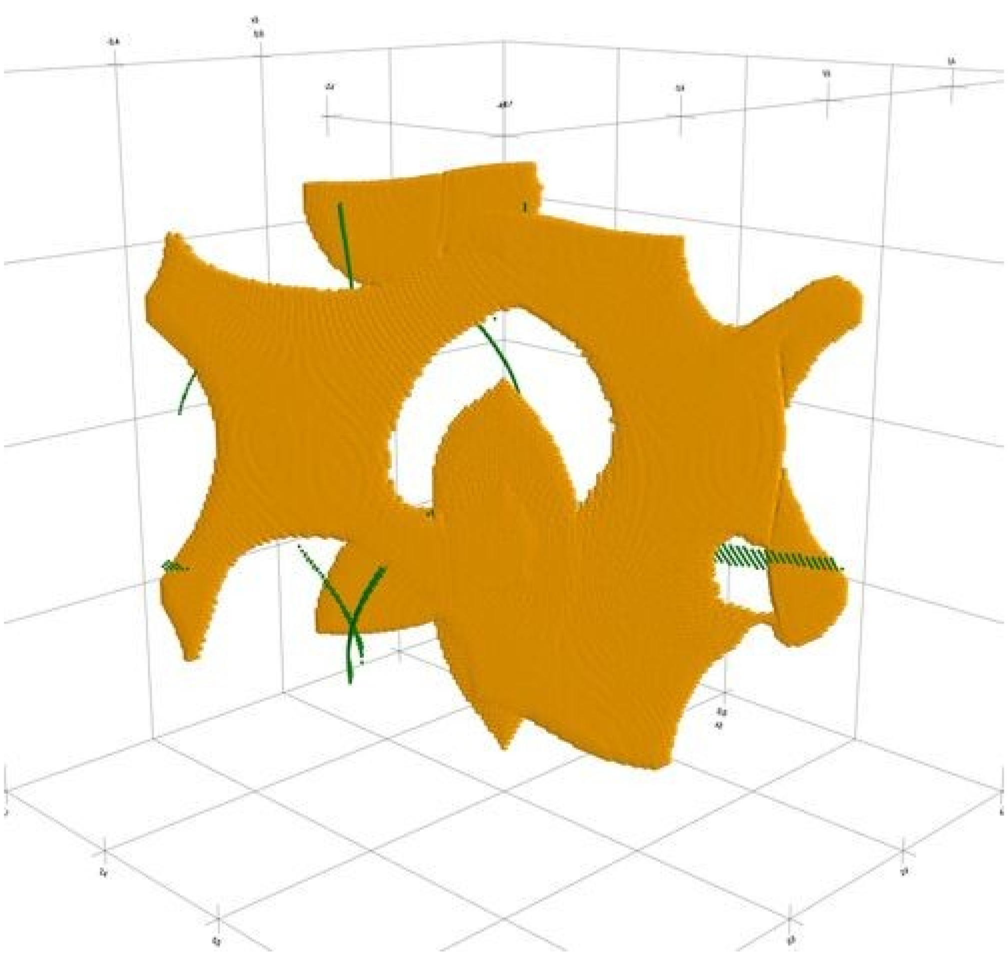
Final positions



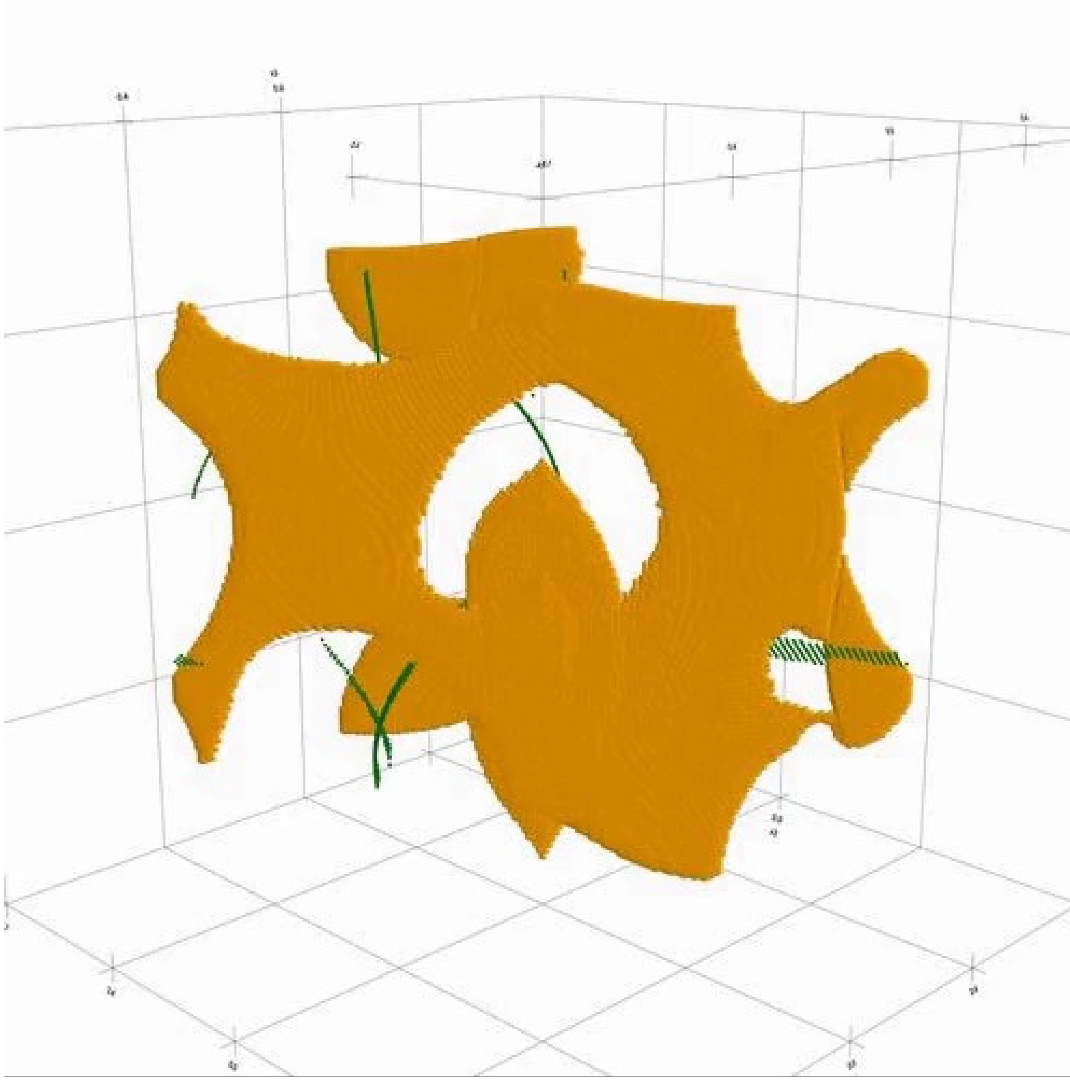
Initial positions



Initial positions



Final positions



1) Context, motivation, goals

2) Turbulence model 1 (effective physical parameters) (‘BxC’)

3) Turbulence model 2 (M << 1, incompressible limit) (‘Muscats’)

4) Turbulence model 3 (M >> 1, compressible limit) (work in progress)

5) Prospects & questionings

Parametric observables/diagnosis tools

Equations of motion (**physical parameters**)

↓

Velocity, density, magnetic fields

↓

Statistical features

↓

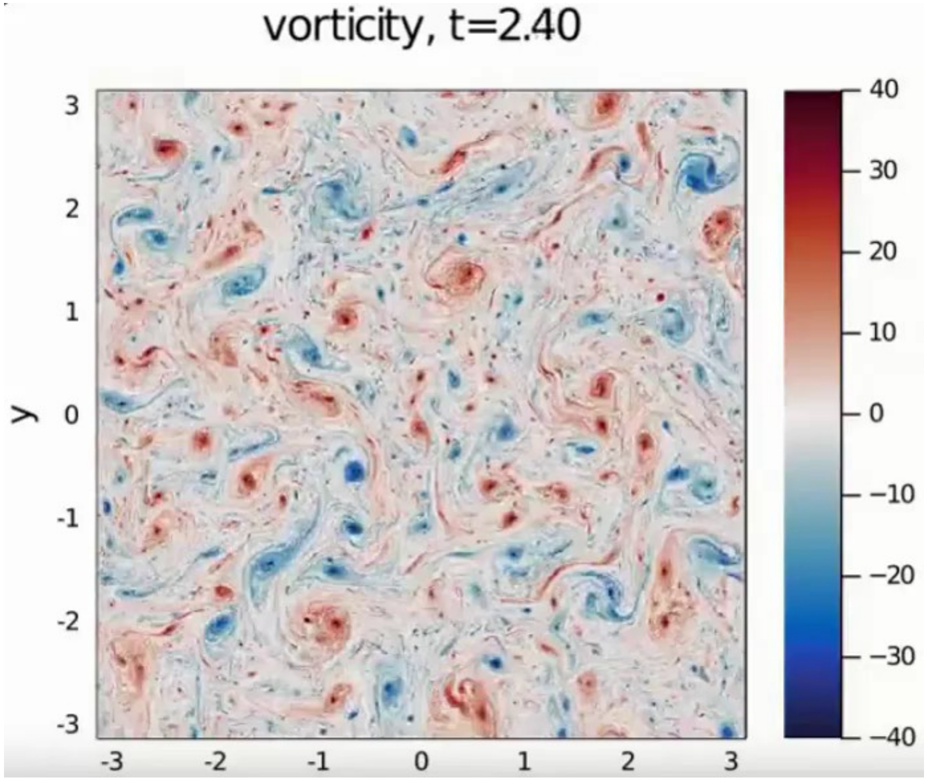
Observations

Parametric observables/diagnosis tools

Link physical parameters to statistics/observables:

Illustration of a method in this direction: Gilbert 88 (JFM)

Imagine a flow like:



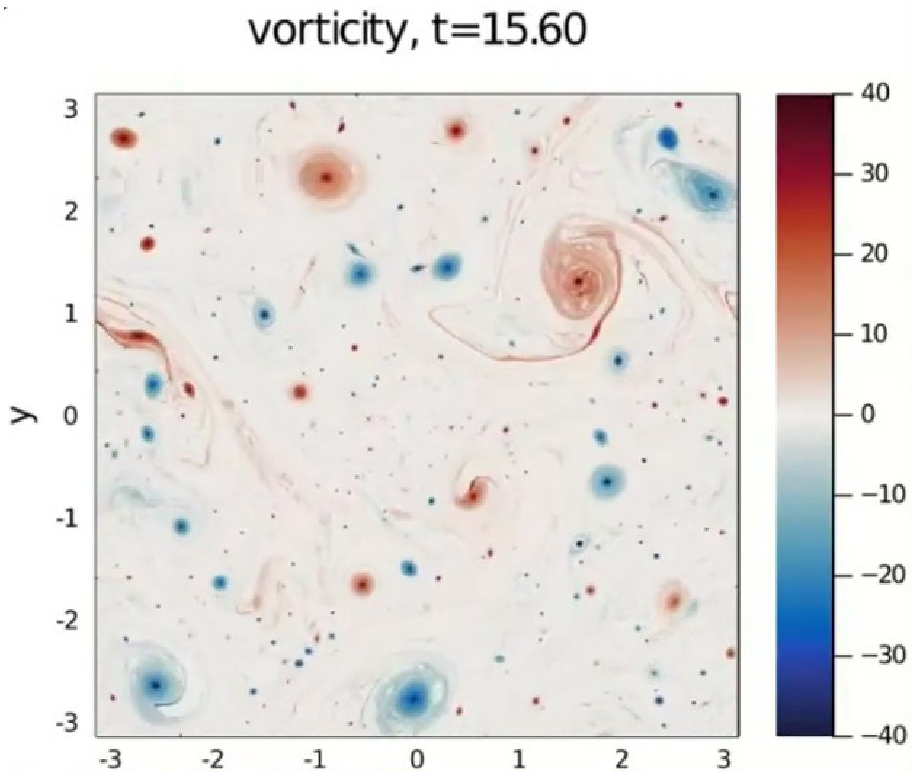
Source: https://www.youtube.com/watch?v=S8iEpFSCZQM

Parametric observables/diagnosis tools

Link physical parameters to statistics/observables:

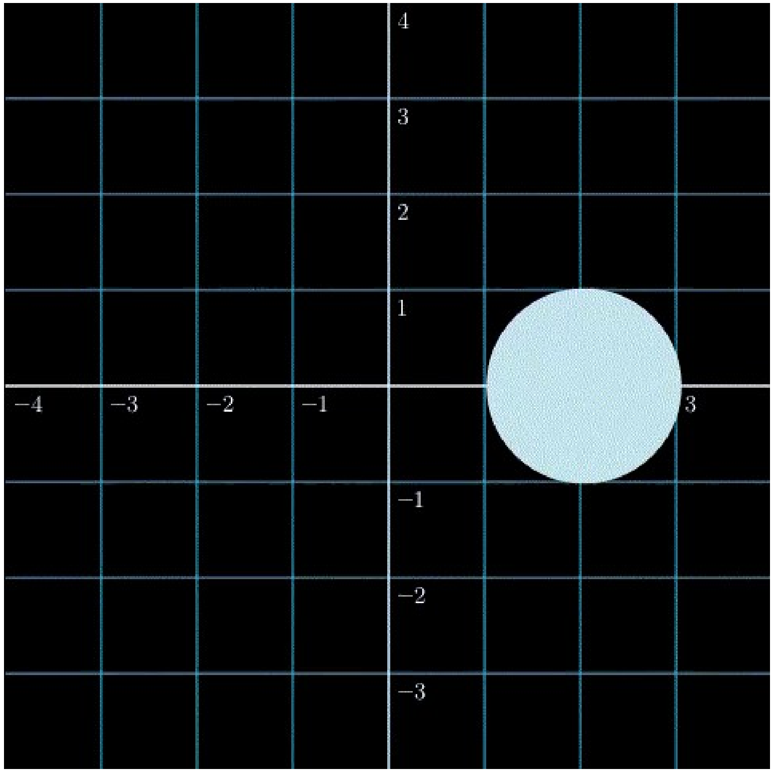
Illustration of a method in this direction: Gilbert 88 (JFM)

Imagine a flow like:



Source: https://www.youtube.com/watch?v=S8iEpFSCZQM

Toy model:

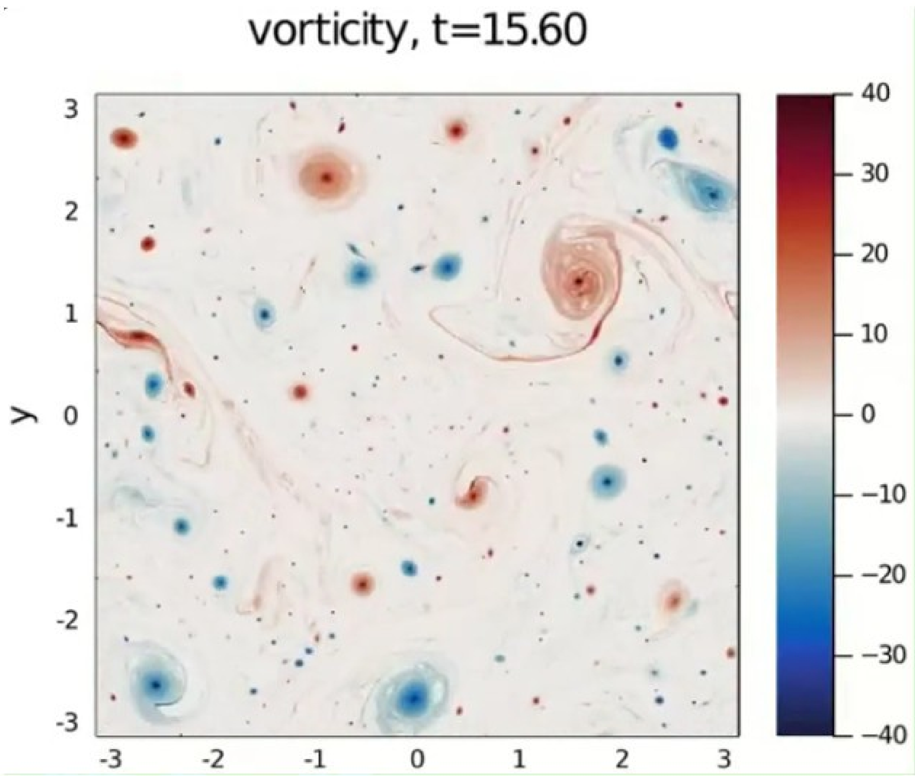


Parametric observables/diagnosis tools

Link physical parameters to statistics/observables:

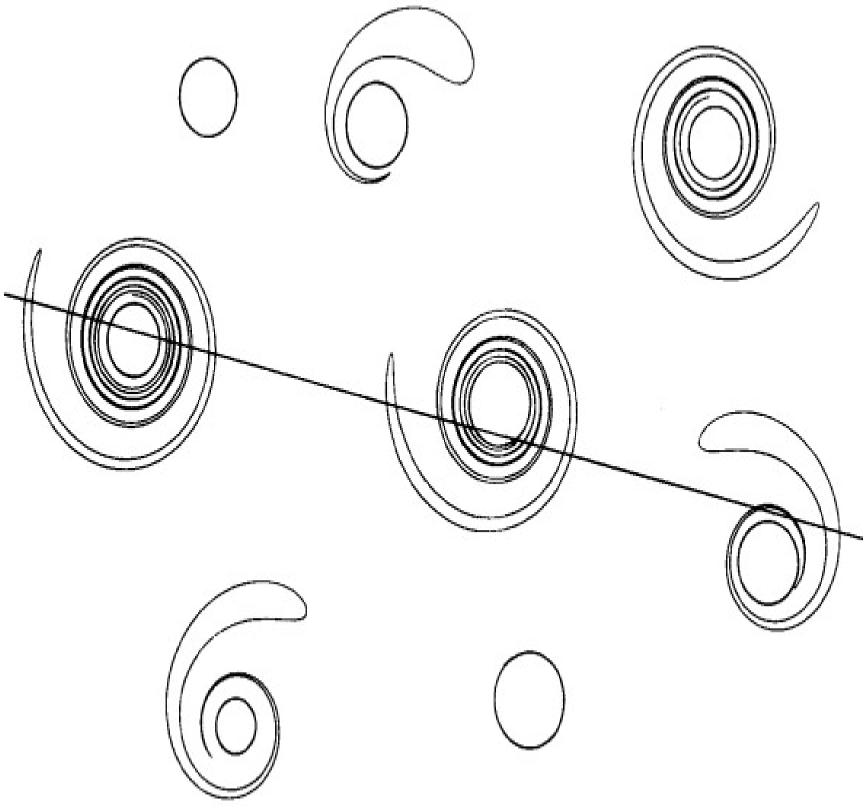
Illustration of a method in this direction: Gilbert 88 (JFM)

Imagine a flow like:

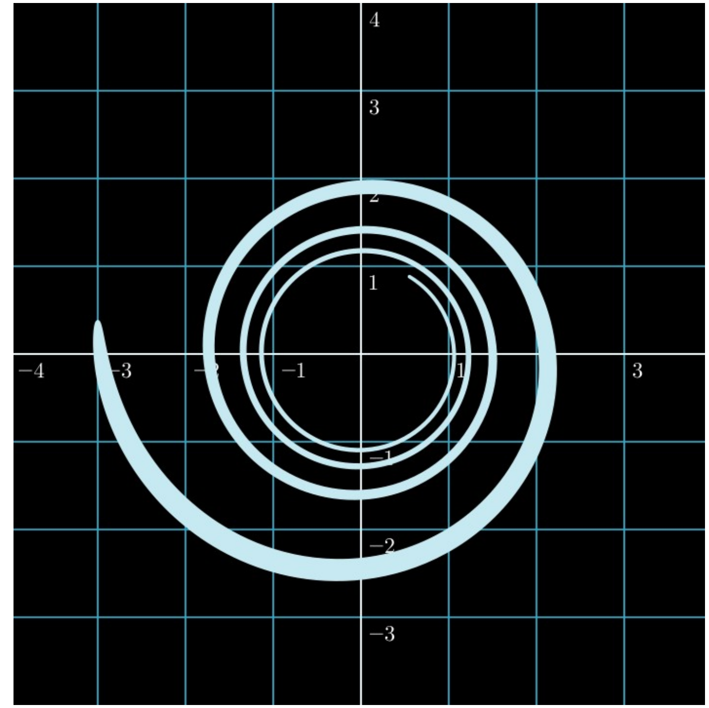


Source: https://www.youtube.com/watch?v=S8iEpFSCZQM

Toy model:



Fourier transform:



$$\hat{w}(k) = \int w(x) e^{-ikx} \mathrm{d}x$$

Here the signal (vorticity) is a sum of Heavisides (series of discontinuities) so:

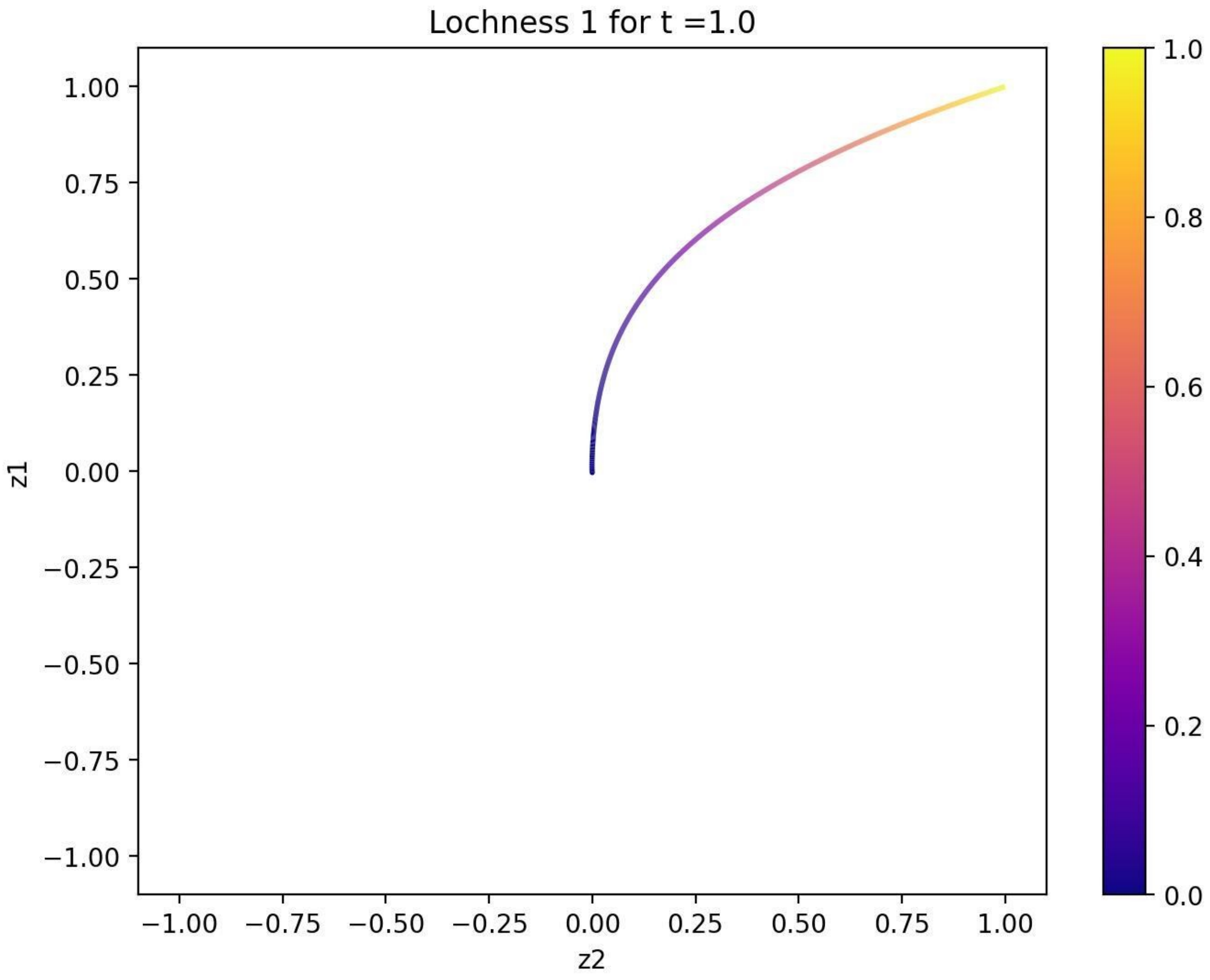
$$\hat{w}(k) = \frac{1}{k} \sum_{n=1}^{N} (-1)^n e^{-ikx_n}$$

‘Exponential sum’ in mathematics

Interlude: Exponential sums

$$z(N) = \sum_{n=1}^{N} e^{2i\pi\phi(n)} \qquad \phi(n) = n^{-0.7} \qquad 1 \leq N \leq 4000$$

Smooth

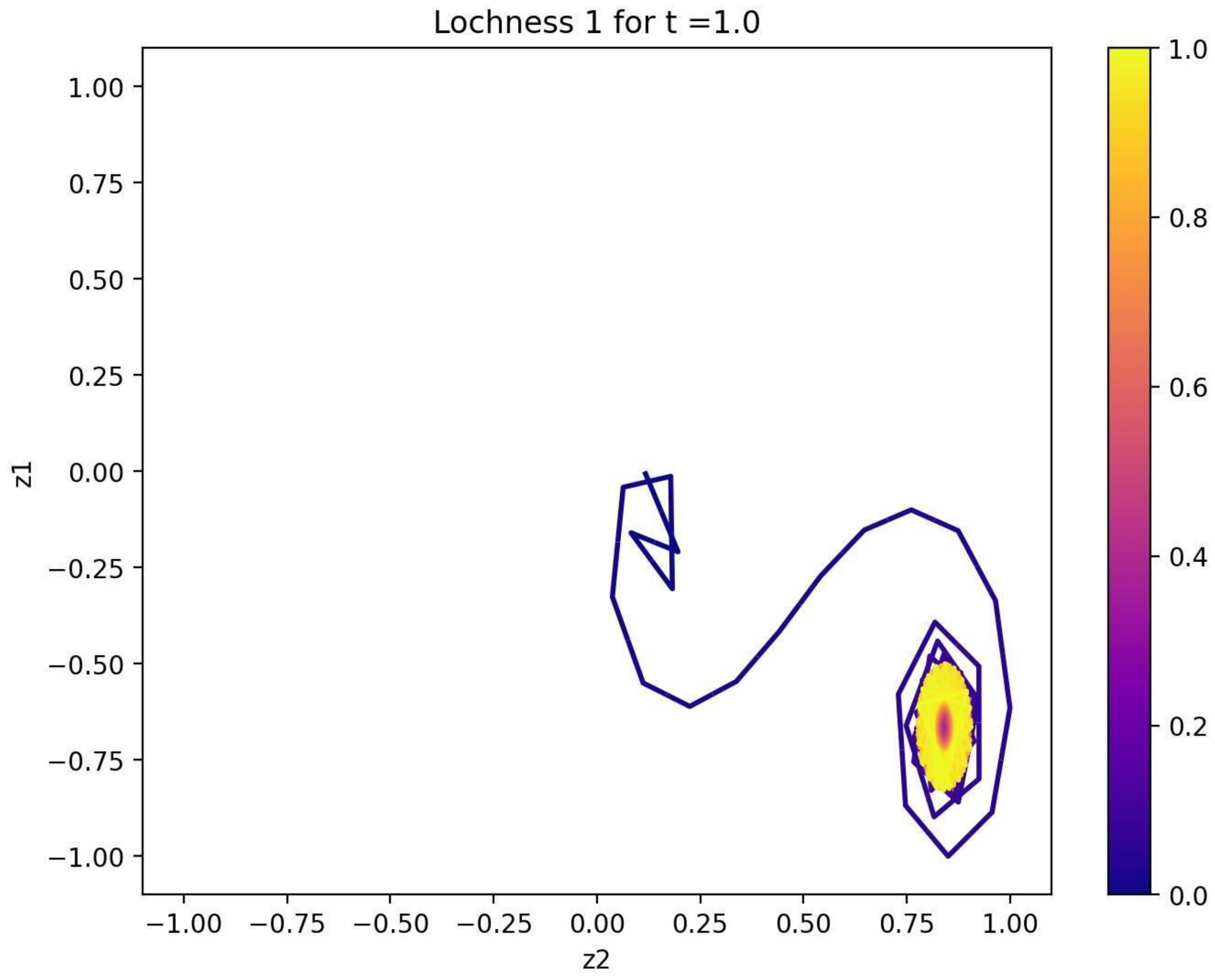


See e.g. https://www.unsw.edu.au/science/our-schools/maths/our-school/spotlight-on-our-people/history-school/glimpses-mathematics-and-statistics/exponential-sums

Interlude: Exponential sums

$$z(N) = \sum_{n=1}^{N} e^{2i\pi\phi(n)} \qquad \phi(n) = 3\,n^{0.7} \qquad 1 \leq N \leq 400$$

Spiraling

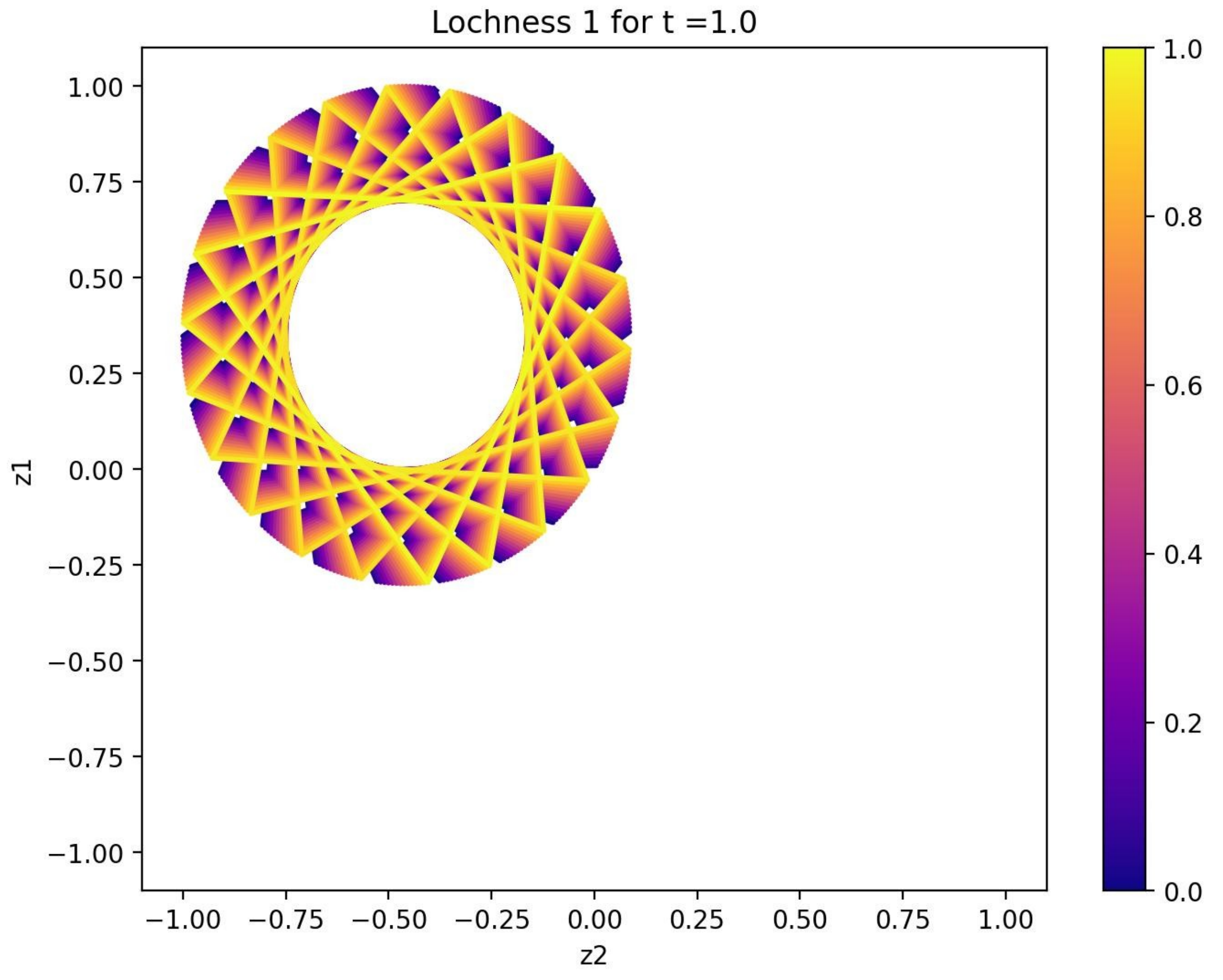


See e.g. https://www.unsw.edu.au/science/our-schools/maths/our-school/spotlight-on-our-people/history-school/glimpses-mathematics-and-statistics/exponential-sums

Interlude: Exponential sums

z(N) = $\sum_{n=1}^{N} e^{2i\pi\phi(n)}$ $\phi(n) = \frac{1}{\pi}n$ $1 \leq N \leq 300$

Bounded

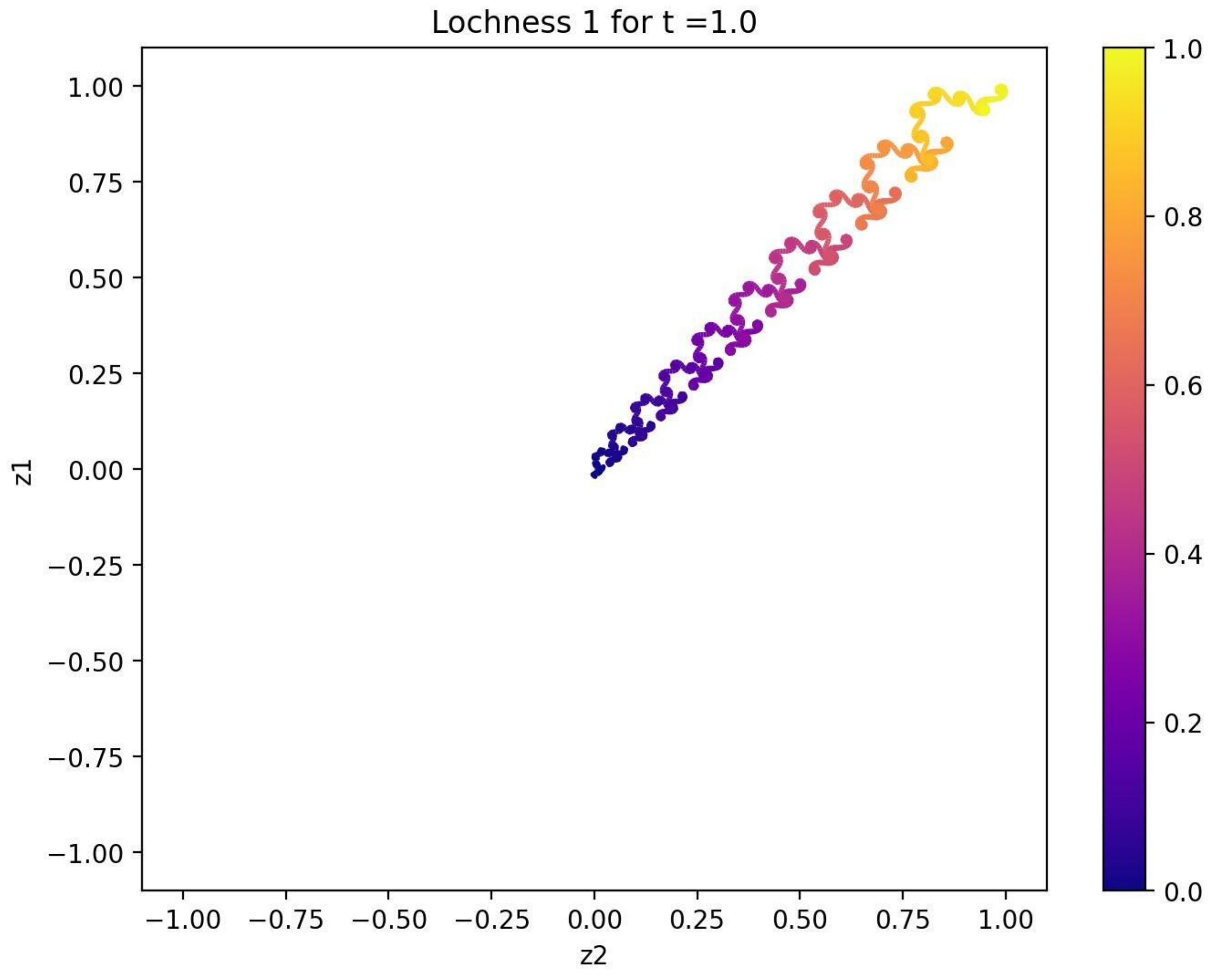


See e.g. https://www.unsw.edu.au/science/our-schools/maths/our-school/spotlight-on-our-people/history-school/glimpses-mathematics-and-statistics/exponential-sums

Interlude: Exponential sums

$$z(N) = \sum_{n=1}^{N} e^{2i\pi\phi(n)} \qquad \phi(n) = n^{1.500} \qquad 1 \leq N \leq 4000$$

Diverging towards
a direction



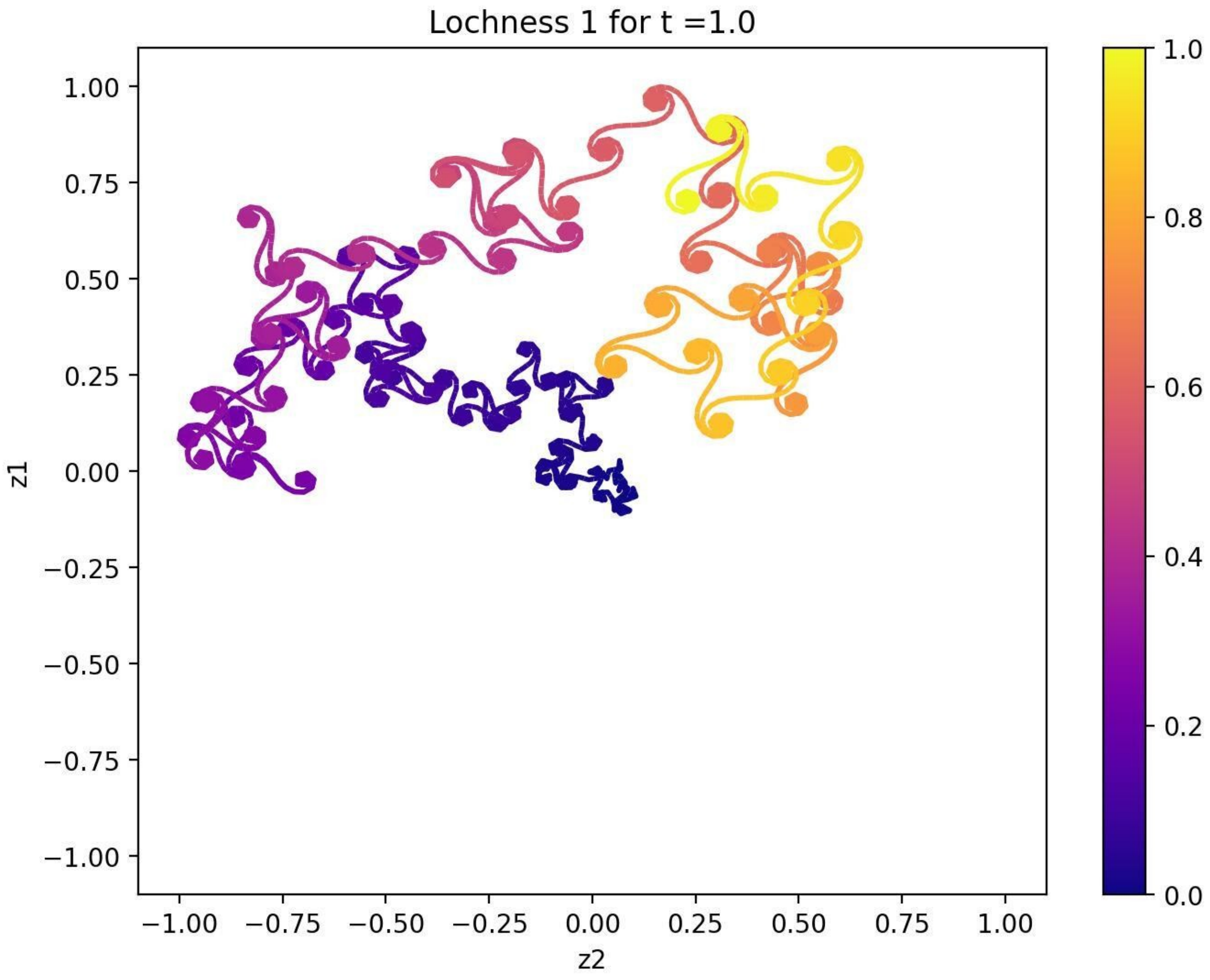
See e.g. https://www.unsw.edu.au/science/our-schools/maths/our-school/spotlight-on-our-people/history-school/glimpses-mathematics-and-statistics/exponential-sums

Interlude: Exponential sums

$$z(N) = \sum_{n=1}^{N} e^{2i\pi\phi(n)} \qquad \phi(n) = n^{1.503} \qquad 1 \leq N \leq 4000$$

Diverging randomly?

High sensitivity!

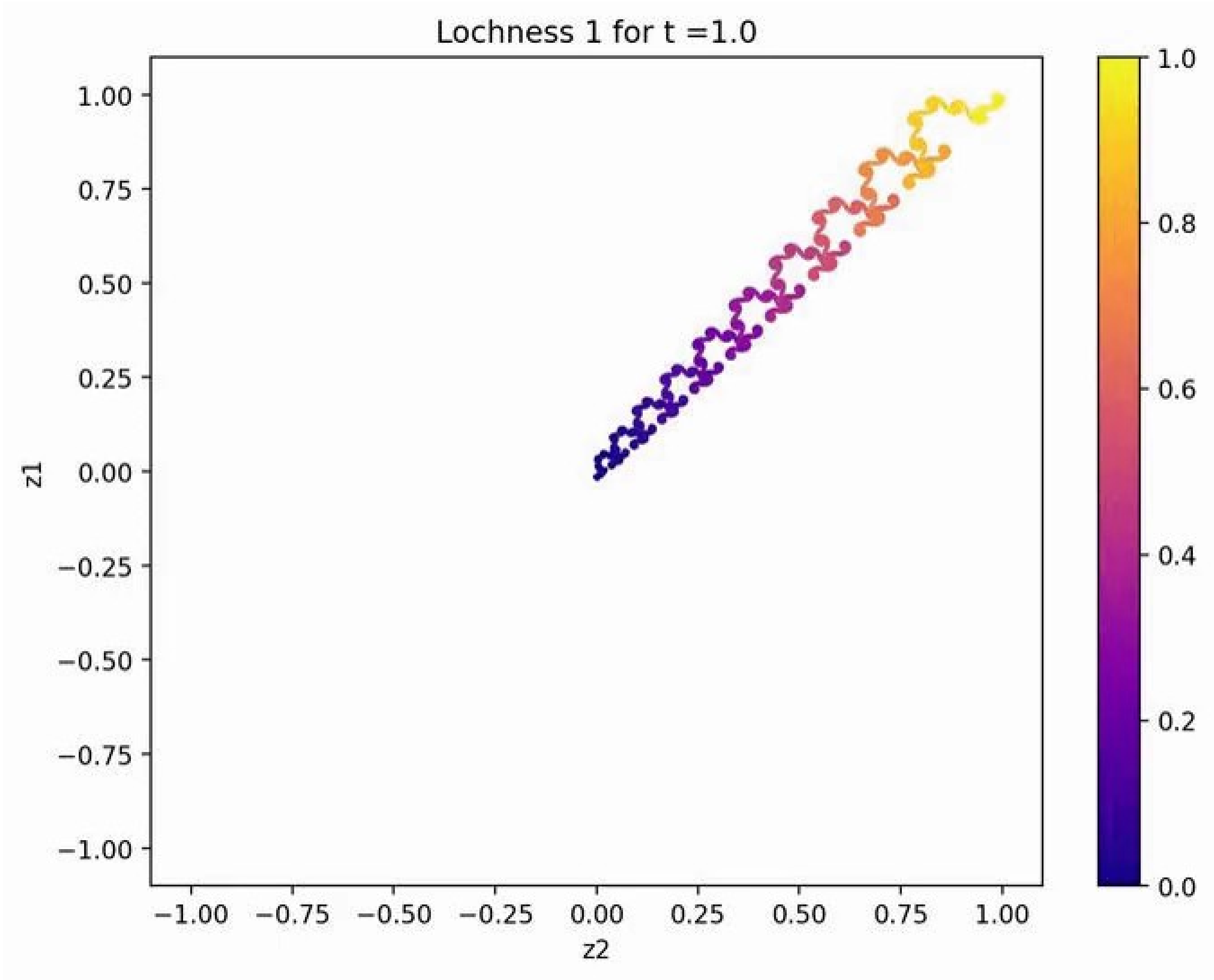


See e.g. https://www.unsw.edu.au/science/our-schools/maths/our-school/spotlight-on-our-people/history-school/glimpses-mathematics-and-statistics/exponential-sums

Interlude: Exponential sums

$$z(N) = \sum_{n=1}^{N} e^{2i\pi\phi(n)} \qquad \phi(n) = n^{1.500 \le \alpha \le 1.503} \qquad 1 \le N \le 4000$$

High sensitivity!

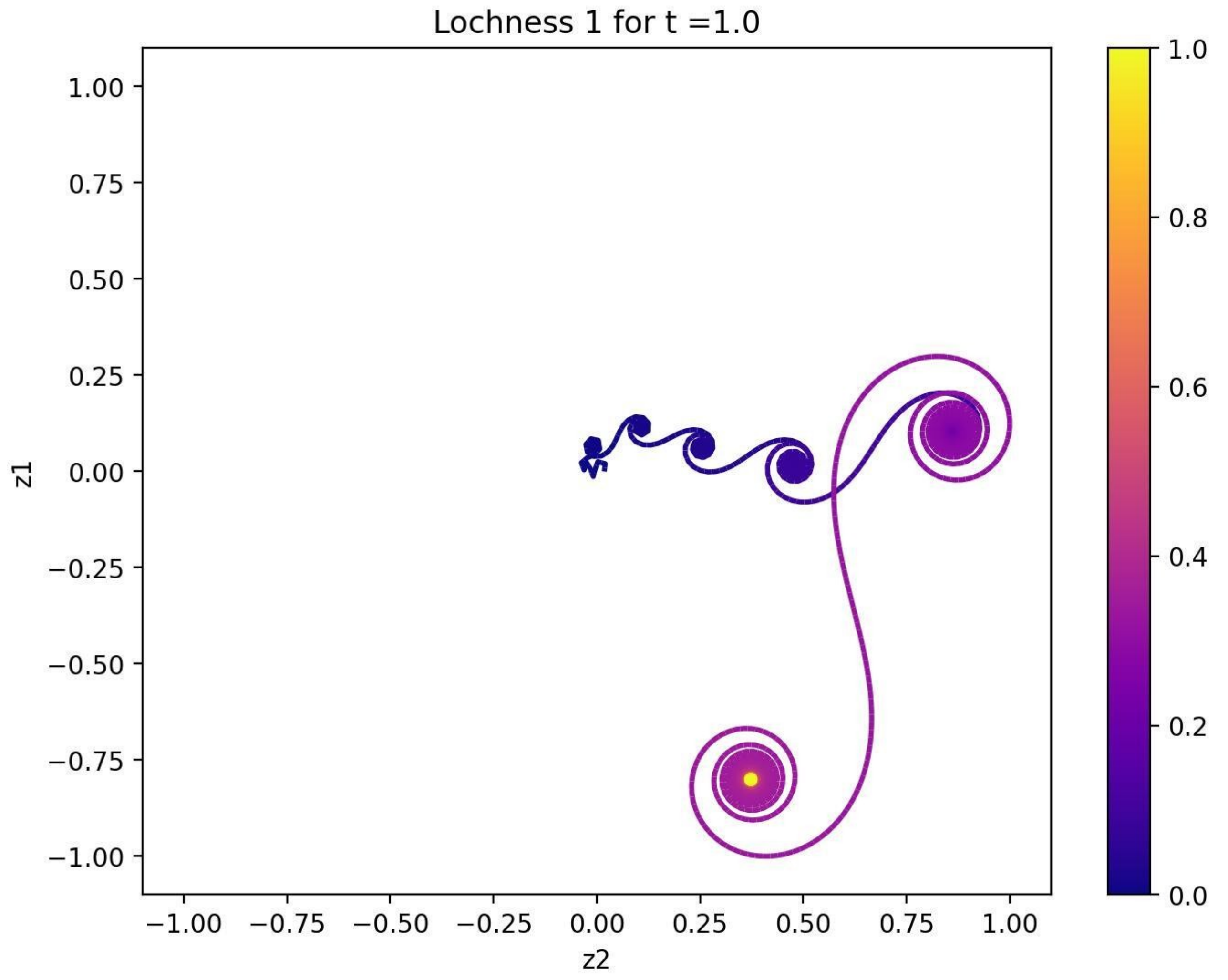


See e.g. https://www.unsw.edu.au/science/our-schools/maths/our-school/spotlight-on-our-people/history-school/glimpses-mathematics-and-statistics/exponential-sums

Interlude: Exponential sums

$$z(N) = \sum_{n=1}^{N} e^{2i\pi\phi(n)} \qquad \phi(n) = \log(n)^4 \qquad 1 \leq N \leq 5000$$

Converging

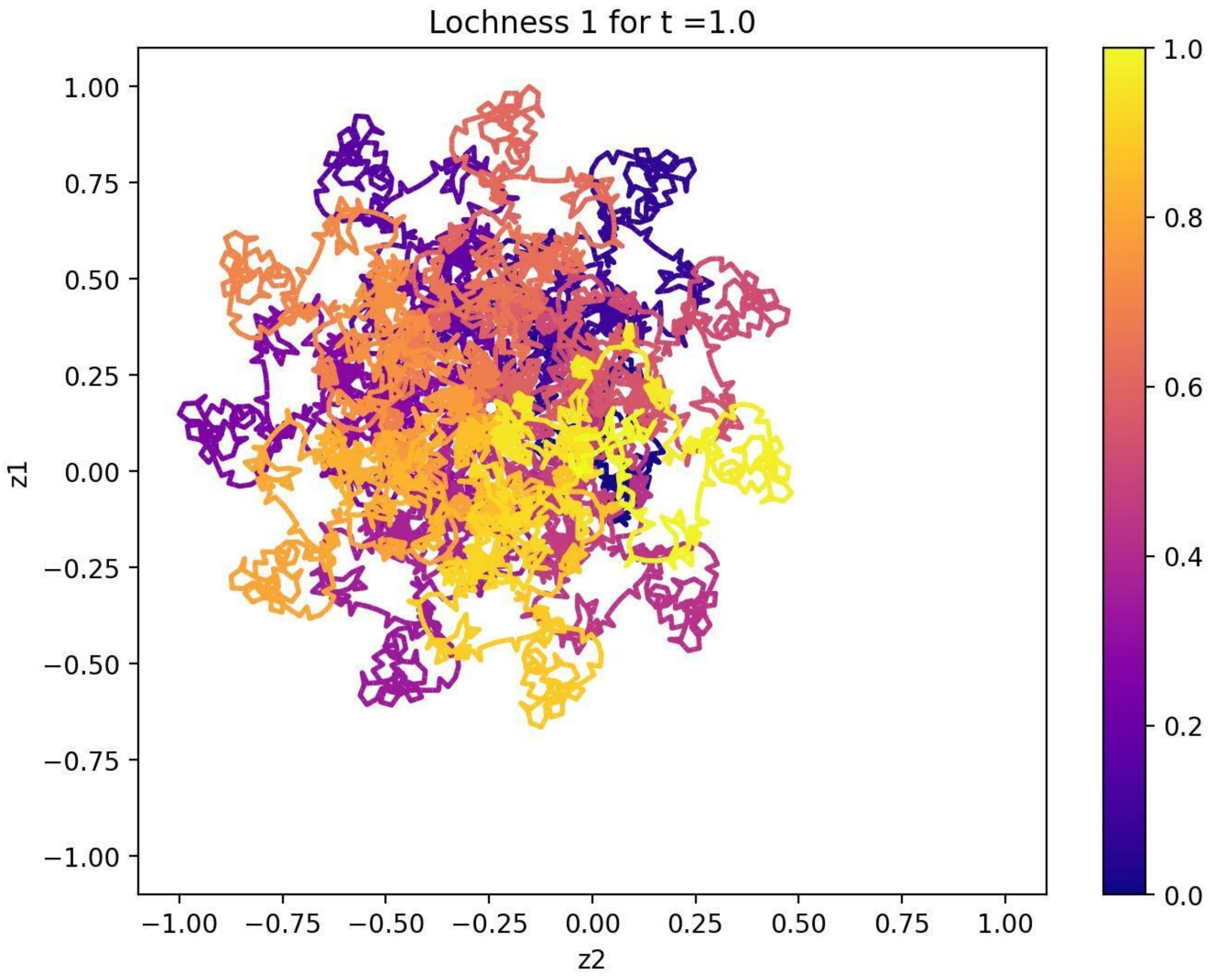


See e.g. https://www.unsw.edu.au/science/our-schools/maths/our-school/spotlight-on-our-people/history-school/glimpses-mathematics-and-statistics/exponential-sums

Interlude: Exponential sums

$$z(N) = \sum_{n=1}^{N} e^{2i\pi\phi(n)} \qquad \phi(n) = n/11 + n^2/21 + n^3/31 \qquad 1 \leq N \leq 7161$$

Highly symmetric

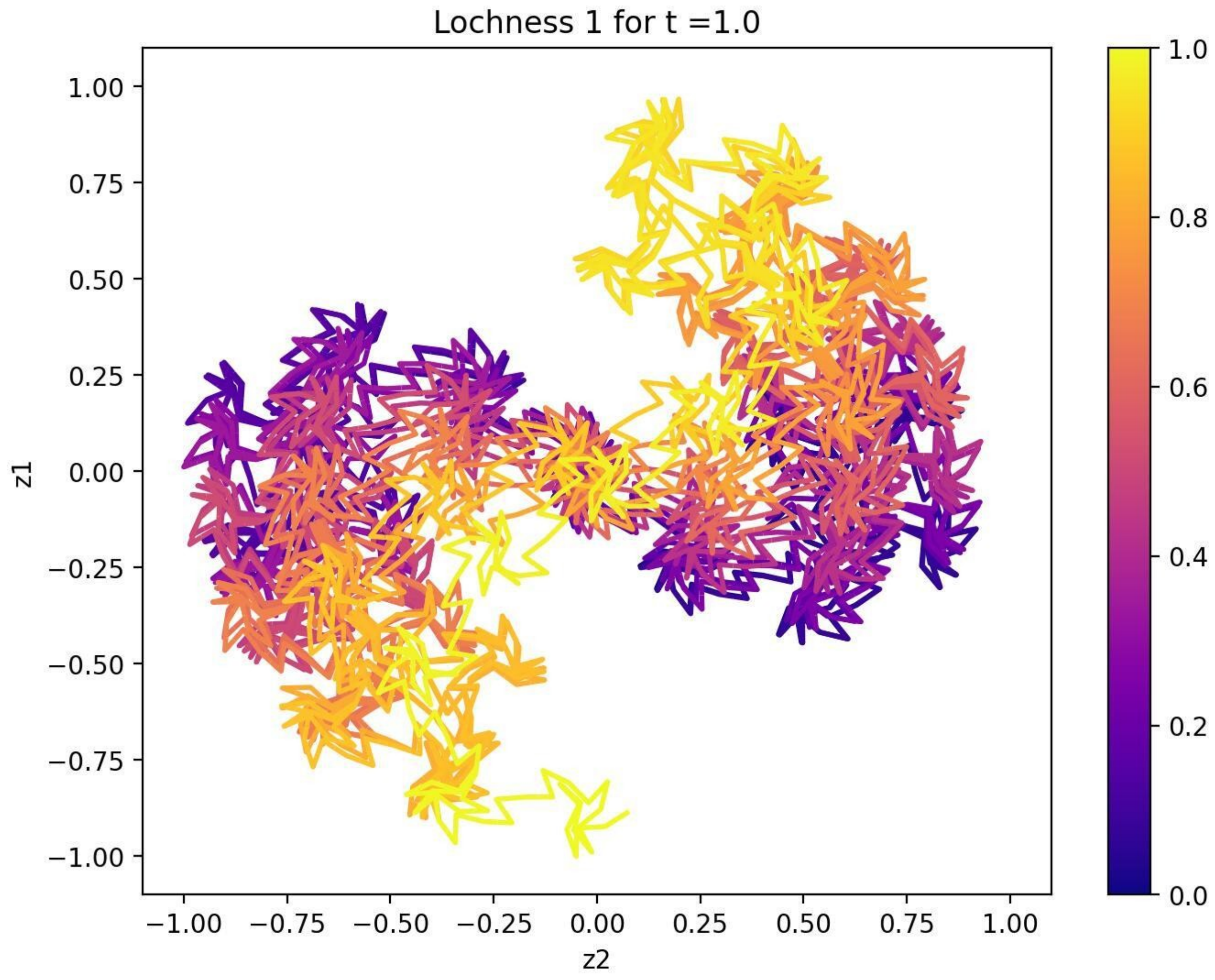


See e.g. https://www.unsw.edu.au/science/our-schools/maths/our-school/spotlight-on-our-people/history-school/glimpses-mathematics-and-statistics/exponential-sums

Interlude: Exponential sums

$$z(N) = \sum_{n=1}^{N} e^{2i\pi\phi(n)} \qquad \phi(n) = \frac{1}{2\pi}n^2 \qquad 1 \leq N \leq 4000$$

Highly symmetric

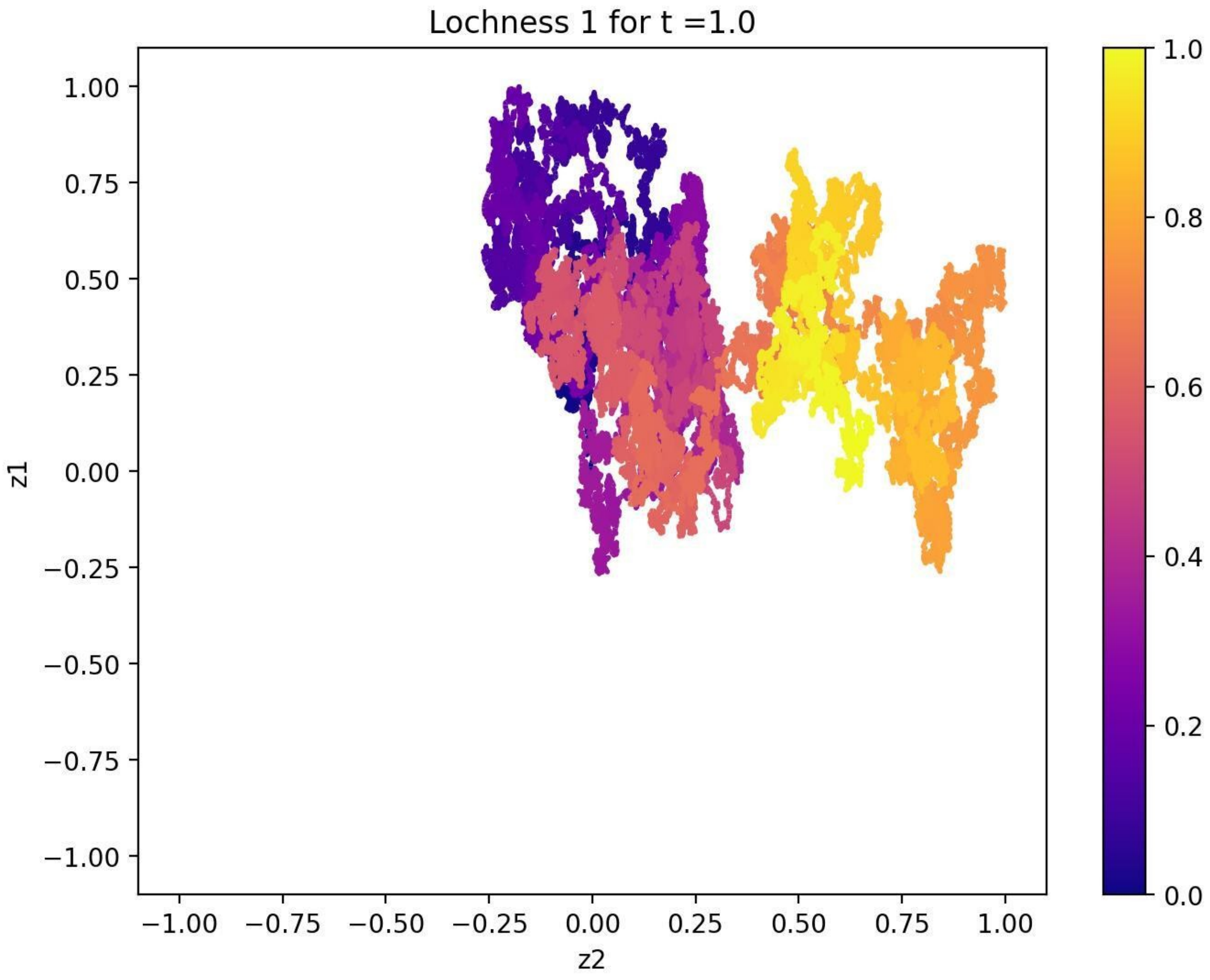


See e.g. https://www.unsw.edu.au/science/our-schools/maths/our-school/spotlight-on-our-people/history-school/glimpses-mathematics-and-statistics/exponential-sums

Interlude: Exponential sums

$$z(N) = \sum_{n=1}^{N} e^{2i\pi\phi(n)} \qquad \phi(n) = n^{2.7} \qquad 1 \leq N \leq 50000$$

Random walk?!

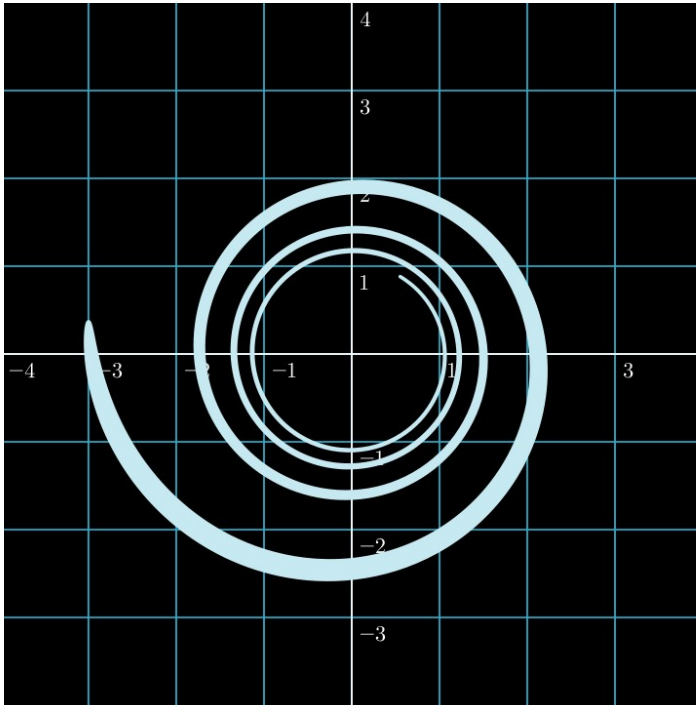


See e.g. https://www.unsw.edu.au/science/our-schools/maths/our-school/spotlight-on-our-people/history-school/glimpses-mathematics-and-statistics/exponential-sums

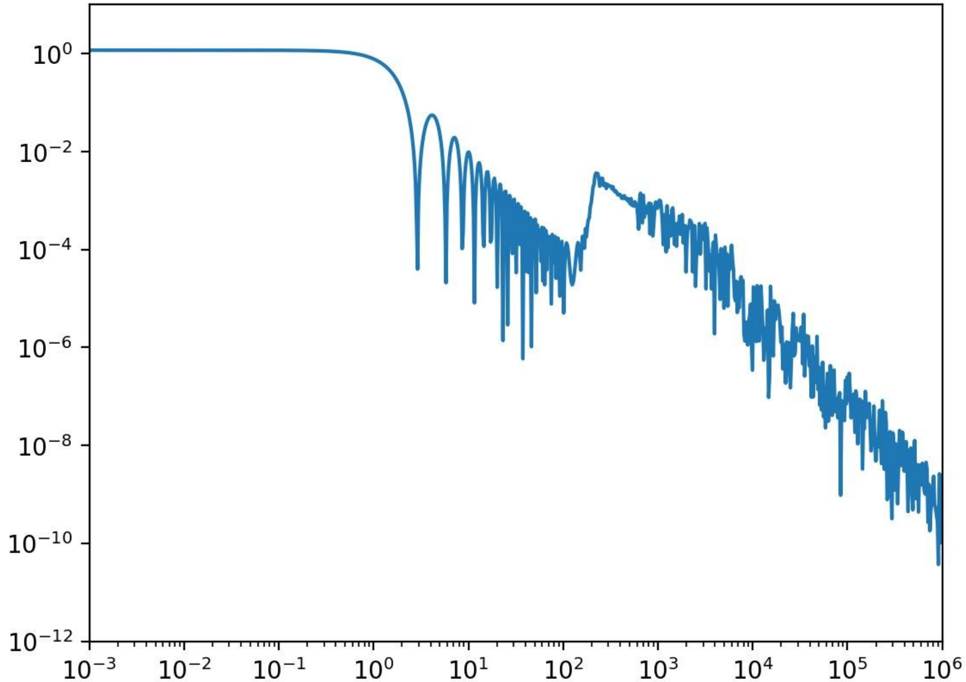
Fourier transform $$\hat{w}(k) = \frac{1}{k}\sum_{n=1}^{N}(-1)^n e^{-ikx_n}$$

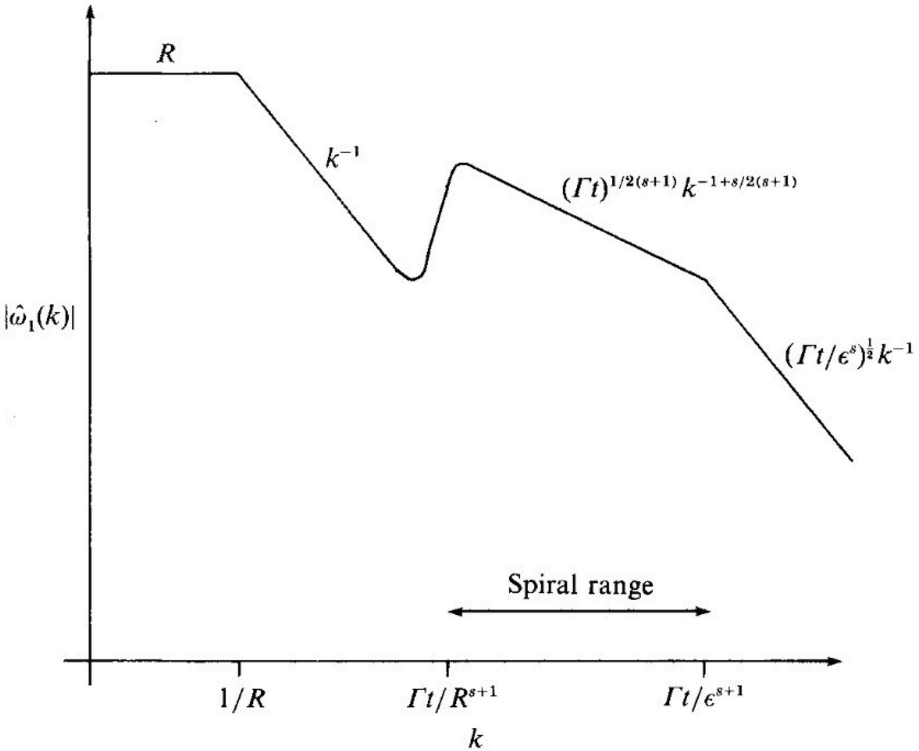
is interpreted as a random walk on the complex plane

→ mean square distance from origin = number of steps



Fourier transform as a function of the physical parameters of the model:

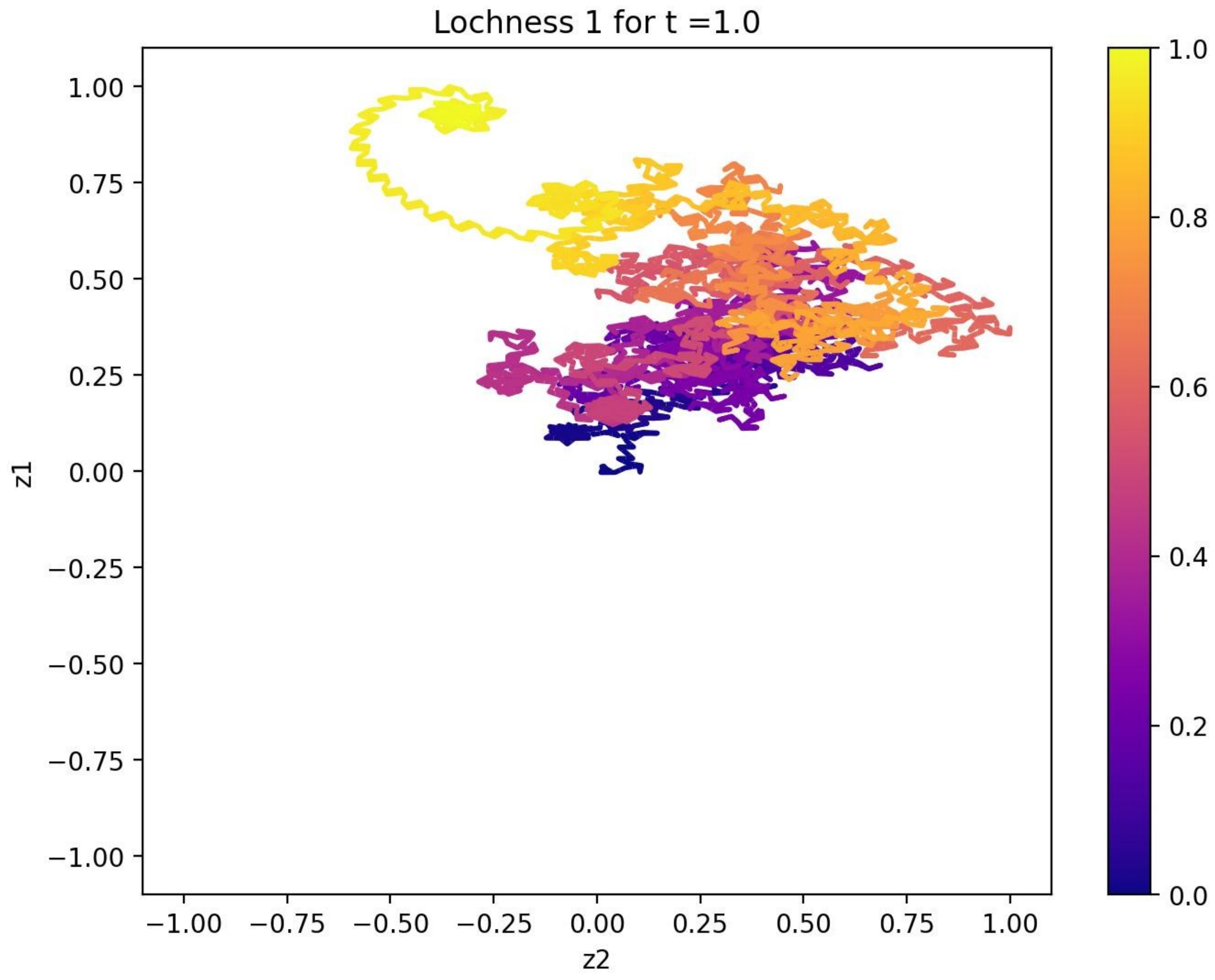




Interlude: Exponential sums (Bonus)

$$z(N) = \sum_{n=1}^{N} e^{2i\pi\phi(n)} \qquad \phi(n) = n^{1.75} \qquad 1 \leq N \leq 4000$$

Intermittency?!

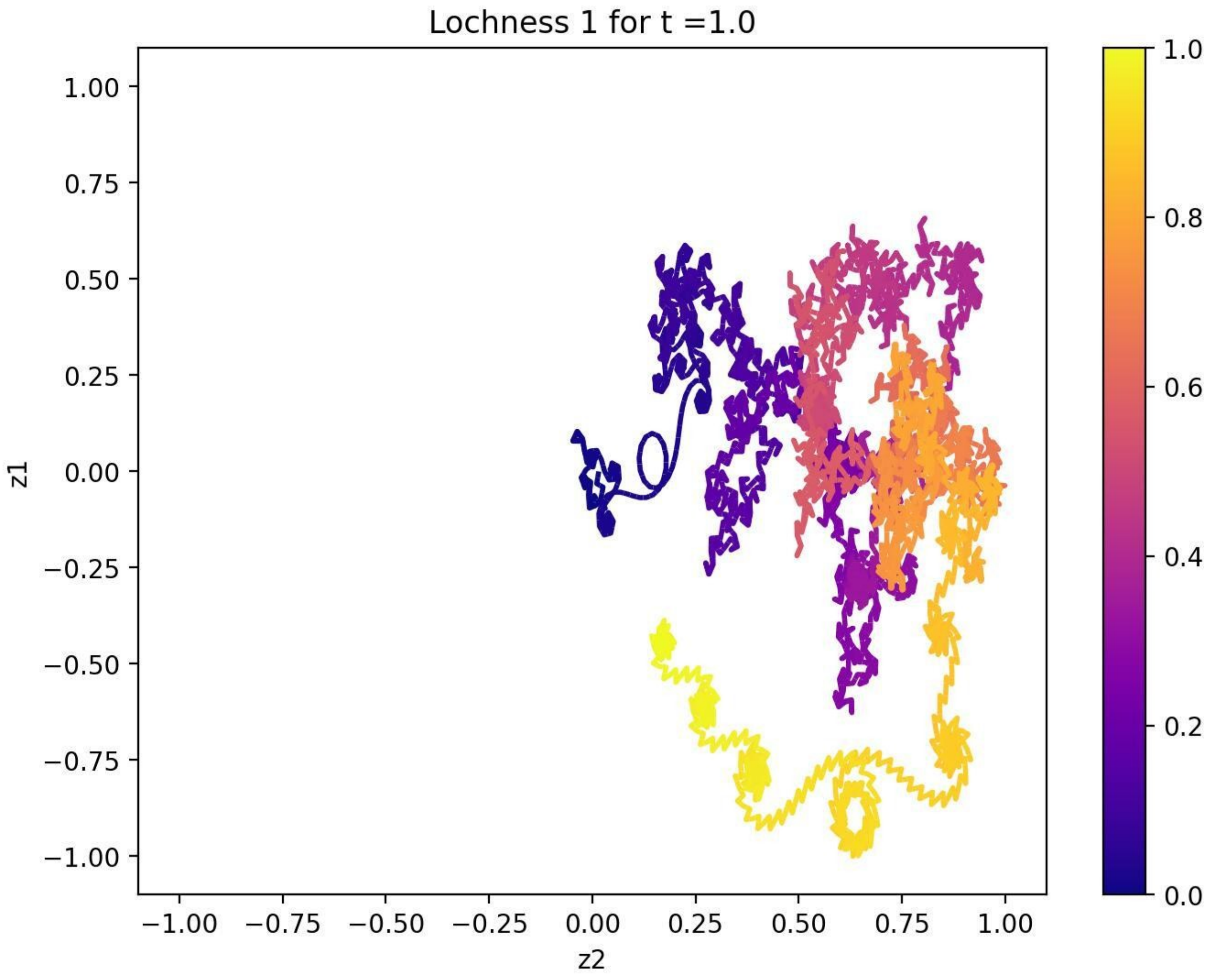


See e.g. https://www.unsw.edu.au/science/our-schools/maths/our-school/spotlight-on-our-people/history-school/glimpses-mathematics-and-statistics/exponential-sums

Interlude: Exponential sums (Bonus)

$$z(N) = \sum_{n=1}^{N} e^{2i\pi\phi(n)} \qquad \phi(n) = n^{1.8877551020408163} \qquad 1 \leq N \leq 4000$$

Intermittency?!

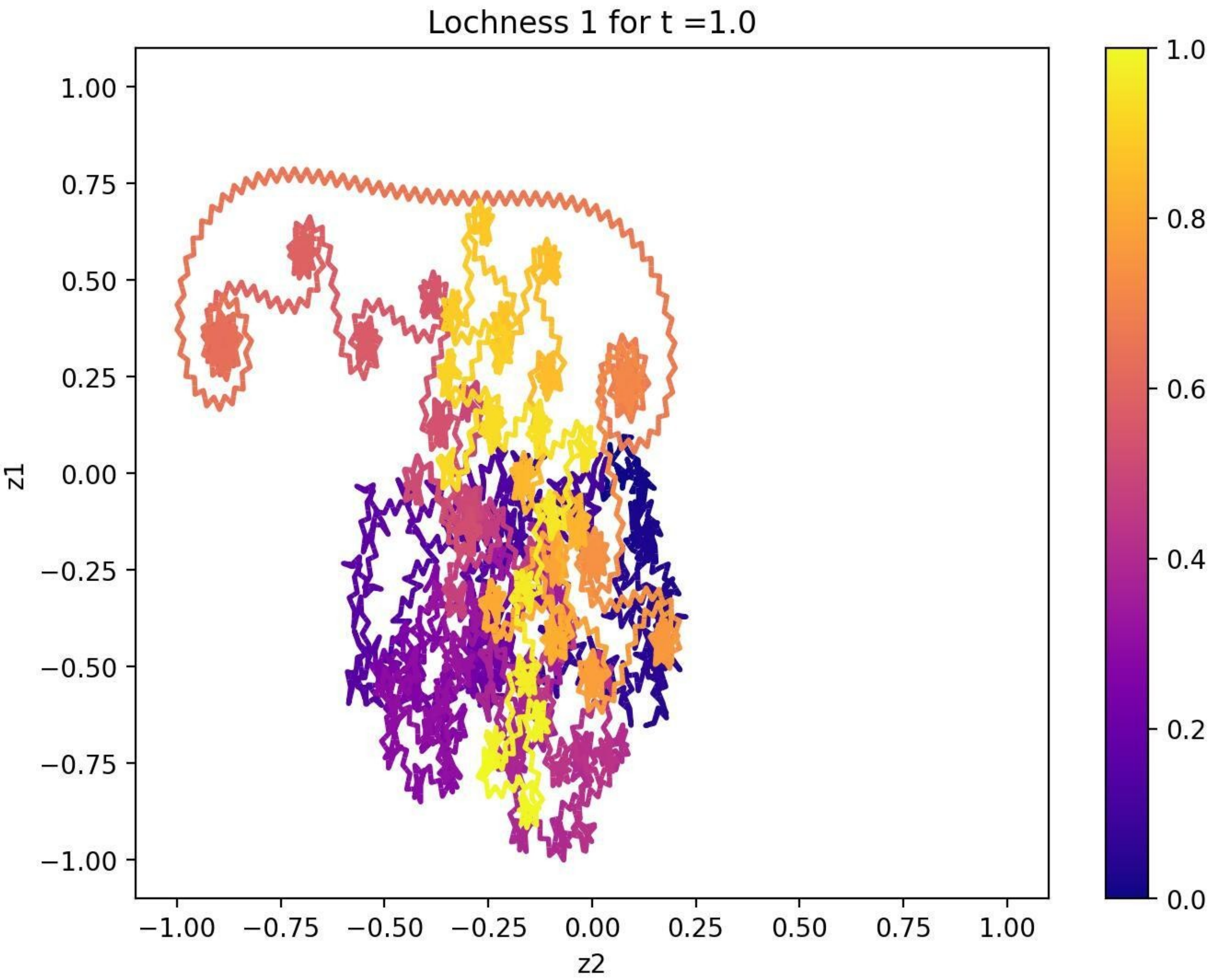


See e.g. https://www.unsw.edu.au/science/our-schools/maths/our-school/spotlight-on-our-people/history-school/glimpses-mathematics-and-statistics/exponential-sums

Interlude: Exponential sums (Bonus)

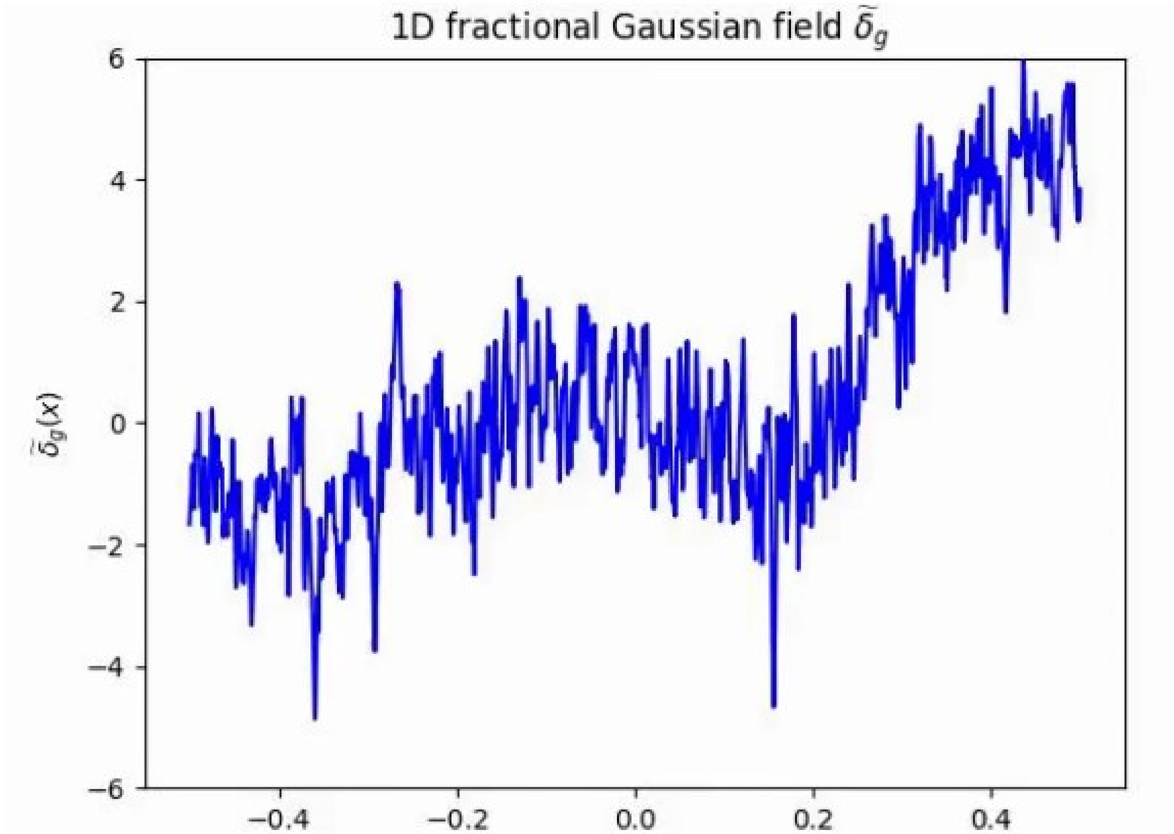
$$z(N) = \sum_{n=1}^{N} e^{2i\pi\phi(n)} \qquad \phi(n) = n^{1.9693877551020407} \qquad 1 \leq N \leq 4000$$

Intermittency?!



See e.g. https://www.unsw.edu.au/science/our-schools/maths/our-school/spotlight-on-our-people/history-school/glimpses-mathematics-and-statistics/exponential-sums

Another important analytical tool complementary to the numerical approach:

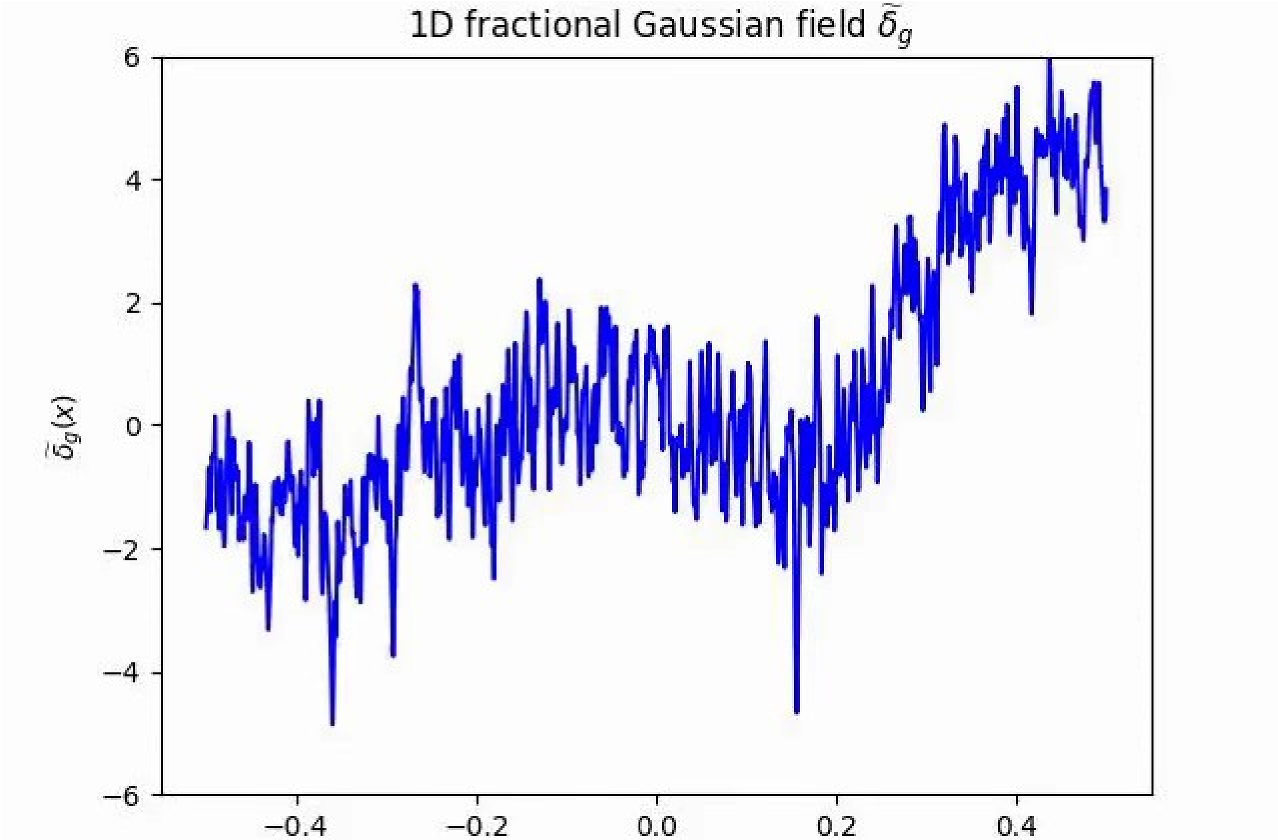


Re >> 1 & M >> 1 => discontinuities/thin structures

=> large dynamical range

=> expensive numerically, but analytically: Distributions!

Another important analytical tool complementary to the numerical approach:

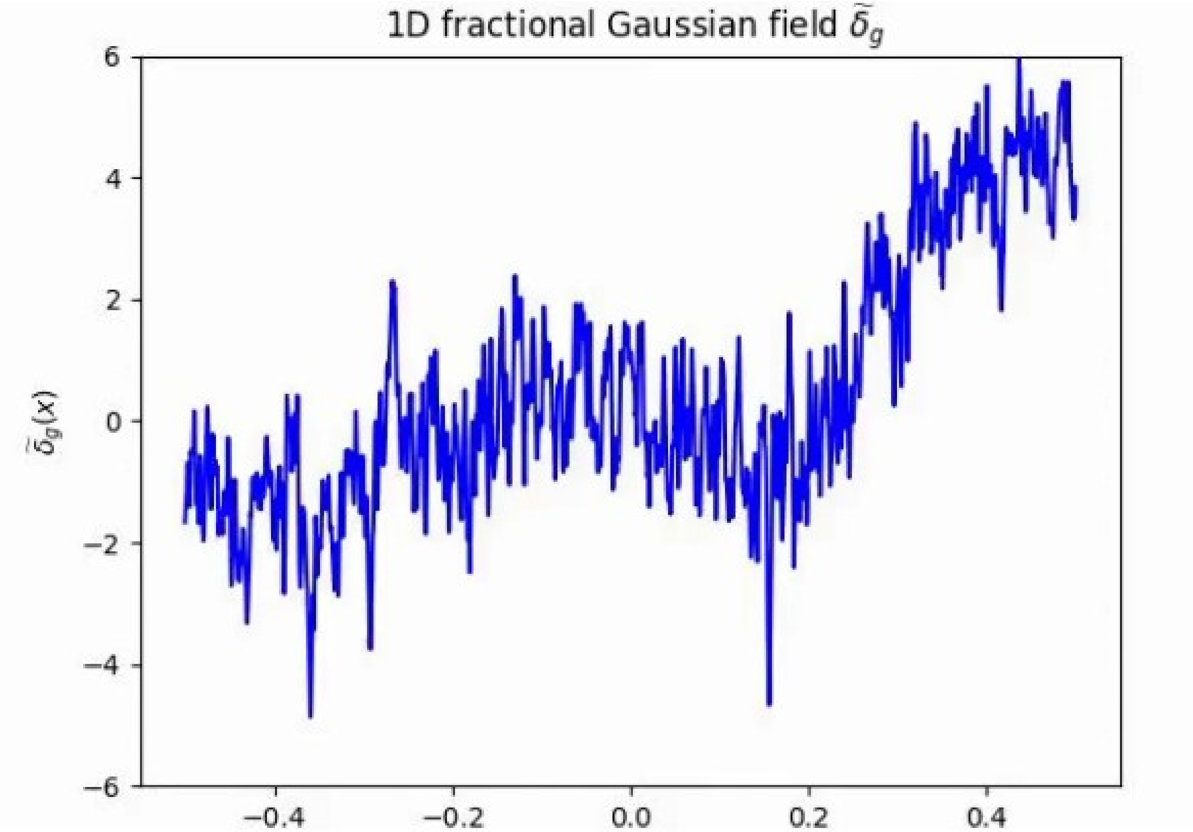


Re >> 1 & M >> 1 => discontinuities/thin structures

=> large dynamical range

=> expensive numerically, but analytically: Distributions!

Another important analytical tool complementary to the numerical approach:

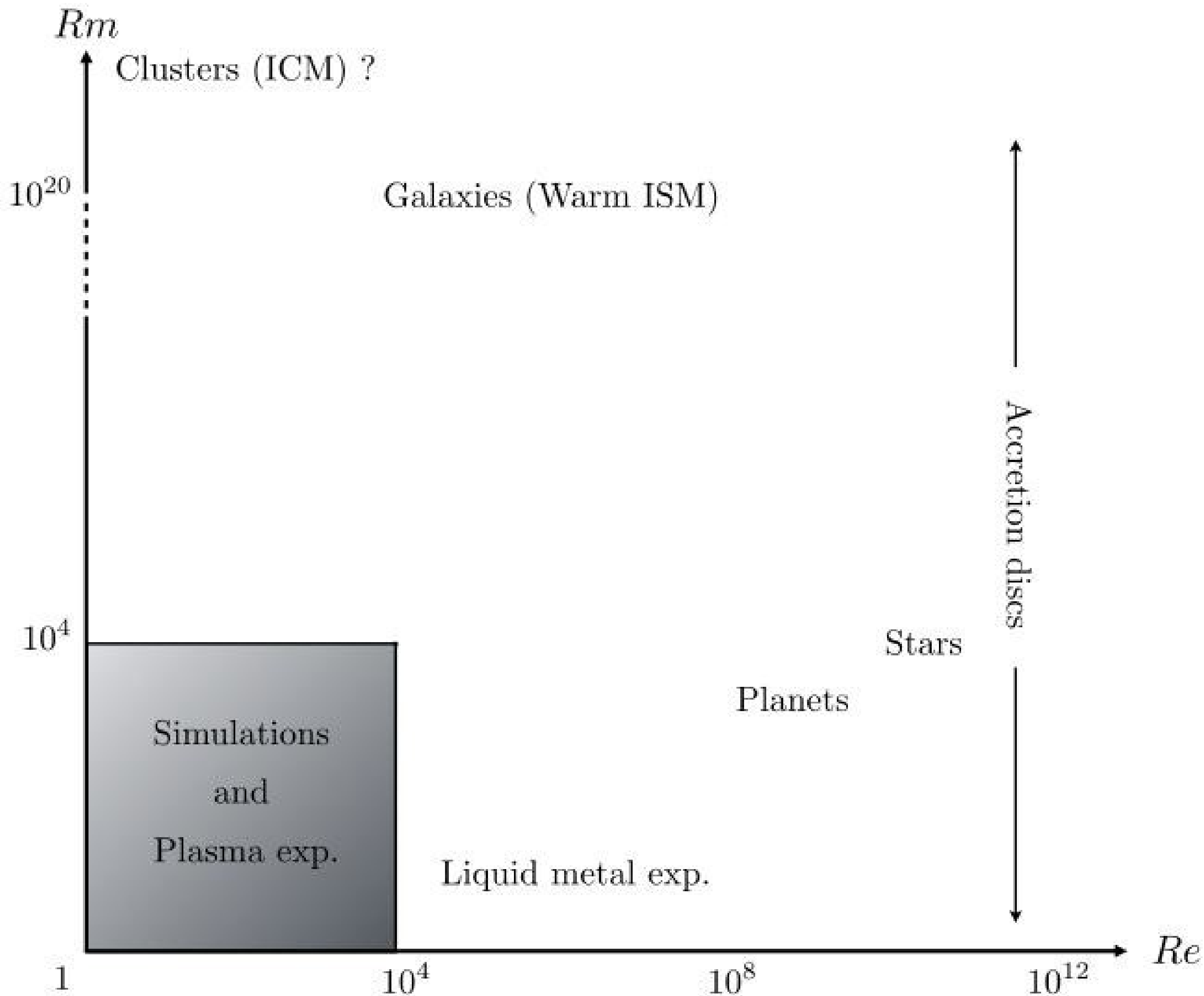


Re >> 1 & M >> 1 => discontinuities/thin structures

=> large dynamical range

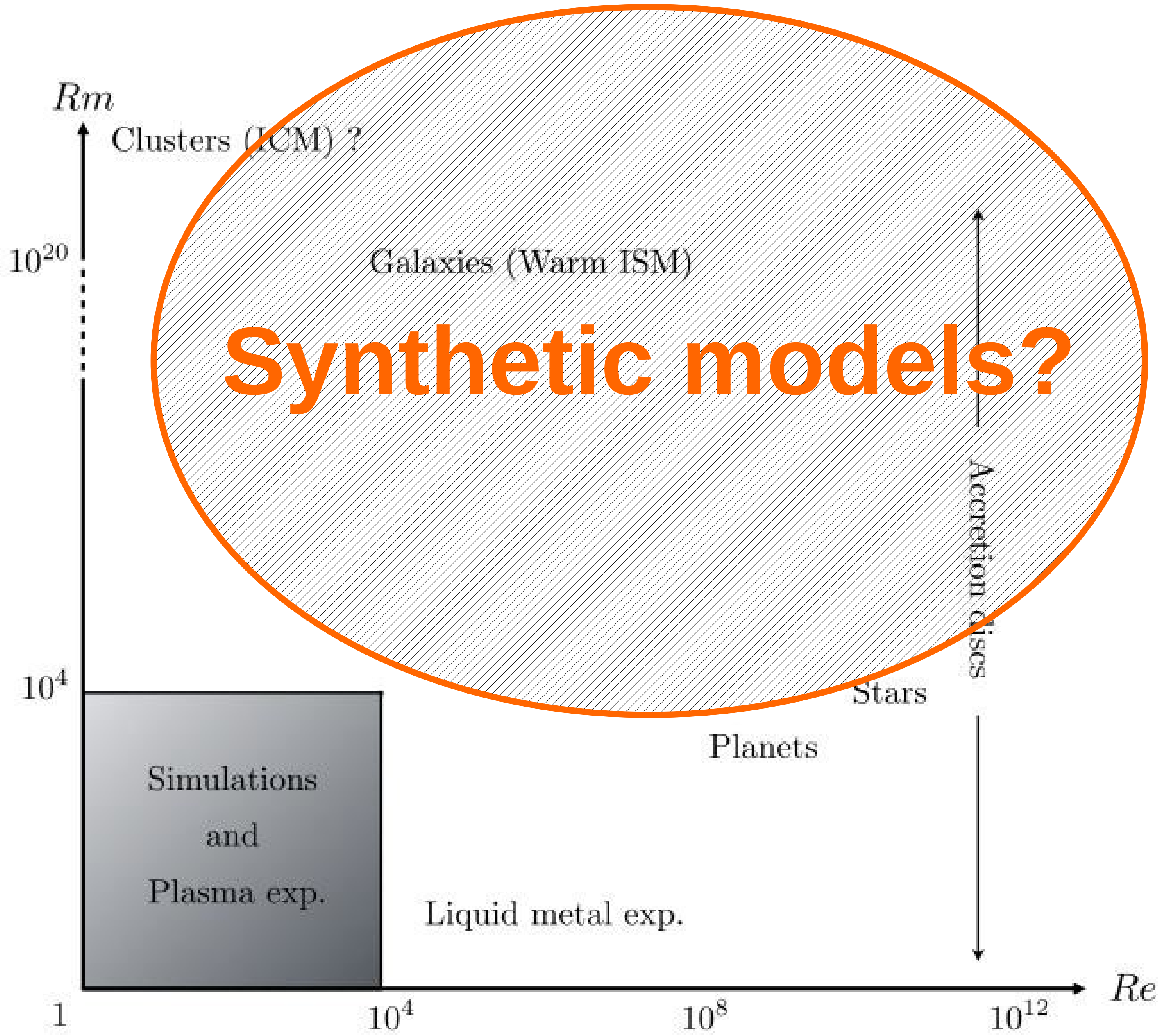
=> expensive numerically, but analytically: Distributions!

Thus we try to model 'numerically unreachable' situations



Rincon, JPP 2019

Thus we try to model 'numerically unreachable' situations



Rincon, JPP 2019

Conclusion

with my **personal** impression & questionings on modeling in Astrophysics:

Astro environments are **insanely complex**: not a fatality but need to be **both** ambitious & humble

to address challenges with adapted pace & tools (e.g. analytical can't do it all, but neither can numerics alone)

Main tools to study this nowadays are numerical (even AI now). Less the case in other fields?

Don't misunderstand me, I am very impressed by numerical work (results and underlying efforts), but I tend to be more attracted by analytical work (preserve the 'old fashioned' approach)

Numerical and analytical work are **complementary**, not in competition, but let's use computers with parsimony. For **environmental** reasons, but even fundamentally: While airplanes exist, they don't replace bicycles & walking. They are unsuited in many situations. Likewise, numerical is not always the most adapted approach.

Analytical turbulence? Even Kolmogorov & Von Neumann called for the numerical approach! Sure. But we are less ambitious on that point. Cf synthetic turbulence.

Thank you for your attention